Axiomatizing Discrete Spatial Relations

Giulia Sindoni[1],, Katsuhiko Sano[2⋆], and John G. Stell[1]

[1] School of Computing, University of Leeds, Leeds, LS2 9JT, UK
{scgsi,j.g.stell}@leeds.ac.uk
[2] Department of Philosophy, Graduate School of Letters, Hokkaido University, Sapporo, Japan
v-sano@let.hokudai.ac.jp

Abstract. Qualitative spatial relations are used in artificial intelligence to model commonsense notions such as regions of space overlapping, touching only at their boundaries, or being separate. Various spatial calculi have been developed including the Region-Connection-Calculus (RCC). The RCC is intended to model topological relations in dense spaces, such as the Euclidean plane, and defines relations using first-order logic. The binary relation of spatial connection is the key primitive in the RCC. In this paper we extend earlier work on qualitative relations in discrete space by presenting a bi-intuitionistic modal logic with universal modalities, called UBiSKt. This logic has a semantics in which formulae are interpreted as subgraphs. We show how a variety of qualitative spatial relations can be defined in UBiSKt. We make essential use of a sound and complete axiomatisation of the logic and an implementation of a tableau based theorem prover to establish novel properties of these spatial relations. We also explore the role of UBiSKt in expressing spatial relations at more than one level of detail. The features of the logic allow it to represent how a subgraph at a detailed level is approximated at a coarser level.

Keywords: Spatial Relations · Discrete Space · Intuitionistic-Modal Logic · Qualitative Representation and Reasoning.

1 Introduction

1.1 The RCC and the Problem of Discrete Space

Qualitative spatial relations are used in artificial intelligence to model commonsense notions such as regions of space overlapping, touching only at their boundaries, or being separate. Qualitative approaches, as opposed to quantitative approaches, abstract from numerical information, which is often unnecessary and even unavailable at the human level. These approaches have become popular in areas like AI and robot navigation, GIS (Geographical Information Systems) and Image Understanding. For a survey on the qualitative representation of spatial knowledge and examples of the problems that can be addressed using these approaches we refer the reader to [4]. Various spatial calculi have been developed including the Region-Connection-Calculus (RCC) [13] and the 9-intersection model [7]. The RCC is a first-order logical theory where a primitive predicate of Connection, C, between regions of the space. From Connection a notion of Parthood is defined by $P(x, y)$ iff $\forall z(C(x, z) \Rightarrow C(y, z))$. Using Connection and Parthood, a set of eight Jointly Exhaustive and Pairwise Disjoint Spatial Relations between regions is obtained. This is known as the RCC-8 in Fig. 1.

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The RCC-8 can distinguish Non-Tangential Proper Part ($NTPP$) from Tangential Proper Part (TPP). RCC-8 can express the relation of sharing only a part, or Partial Overlapping (PO), as well as connection on the boundaries, or External Connection (EC) as well as disjointness, or Disconnection (DC). Equality (EQ), and the inverses of TPP and $NTPP$ are also included.

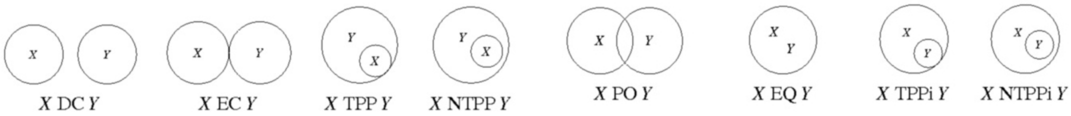


Fig. 1. The RCC-8 Spatial Relations.

Cohn and Varzi in [5] show that the RCC can be interpreted in a topological space where regions of the space are certain non-empty subsets of the space, and an operator of Kuratowski closure c is used to define Connection. Three notions of connection are proposed there:

1. $C_1(x, y) \Leftrightarrow x \cap y \neq \varnothing$
2. $C_2(x, y) \Leftrightarrow c(x) \cap y \neq \varnothing$ or $x \cap c(y) \neq \varnothing$
3. $C_3(x, y) \Leftrightarrow c(x) \cap c(y) \neq \varnothing$

Although the RCC aims to be neutral about whether space is dense, that is whether space can be repeatedly sub-divided *ad infinitum*, it is well known that, once atomic regions are allowed, giving non-empty regions of the space without any proper parts, the RCC theory becomes contradictory [13]. Moreover, as noticed in [18], the use of Kuratowski topological closure prevents the expression of a natural form of connection in a discrete space even in some simple examples.

The ability to reason about discrete space is important in many fields and applications. Any kind of transport network (road networks, railway networks, airlines network) is naturally represented by discrete structures such as graphs. Images in image processing lie in discrete spaces of pixel arrays. Geographical data represented digitally are essentially discrete both in vector and raster formats as ultimately there is a limit to resolution.

1.2 Related work

Galton [8] studied a notion of connection between subsets of a particular kind of discrete space, known as Adjacency Space. This is a set N together with a relation of adjacency $\alpha \subseteq N \times N$. N can be thought of as a set of pixels, following the approach of Rosenfeld's Digital Topology [15]. A single pixel is the atomic region of the space. The relation α is symmetric and reflexive, but not necessarily transitive. Connection, C_α, is defined for subsets $X, Y \subseteq N$ by $C_\alpha(X, Y)$ if there are $a \in X$ and $b \in Y$ such that $(a, b) \in \alpha$. From this eight spatial relations between regions are obtained in a first-order logical theory, a discrete version of the RCC-8, and employed in the context of orrection of segmentation errors in histological images [14].

Galton's discrete space (N, α) can be regarded as a graph where N is the set of nodes and the relation $\alpha \subset N \times N$ gives the edges. There is a notable difference between theory of adjacency space and graph theory [9]. A substructure of an adjacency space can be specified just in terms of nodes, two nodes being connected by only one edge, or relation of adjacency. This is not true in the general setting of a multigraph, where multiple edges may occur between two nodes, and, therefore, different subgraphs sharing the same set of nodes may be considered. Cousty et al. [6] argue that edges need to play a more central role, and make the key observation that sets of nodes which differ only in their edges need to be regarded as distinct. This generality appears also important in examples such as needing to model two distinct roads between the same endpoints, or distinct rail connections between the same two stations. The logic used in the present paper has a semantics in which formulae are interpreted as down-closed sets arising from a set U together with a pre-order H. A special case of a set with a pre-order is a graph, and formulae are interpreted as its subgraphs. Discrete regions are seen as subgraphs and they are more general than Galton's construction. We allow graphs to have multiple edges between the same pair of nodes, thus using a structure sometime called a multi-graph.

The present paper extends the work in [18] where spatial relations between discrete regions are expressed in a logic, which here we denote **UBiSKt**. In [18] the justification of these definitions is entirely semantic. Such an approach validates the expression of spatial relations but does not address reasoning about discrete space in a practical way. The novelty of the present paper is that we provide a sound, complete and decidable axiomatization of the logic **UBiSKt** and that we use this axiomatization to establish several new properties of the discrete spatial relations. In addition, we also present a tableau calculus, extending that in [19], thus providing a computational tool for performing discrete spatial reasoning, which we have proved to be equivalent to the axiomatic proof-system. We begin here to use these new tools by exploring the role of **UBiSKt** in expressing spatial relations at more than one level of detail. The features of the logic and its connection with mathematical morphology are essential in this. This opens many directions for future work, including studying the different notions of approximation expressible in the logic, and being able to reason with spatial relations between regions at different levels of detail.

The paper is structured as follows. Section 2 introduces **UBiSKt**, a bi-intuitionistic modal logic with universal modalities, and a sound and complete axiomatization is given. Section 3 presents **UBiSKt** as a logic for graphs and shows a set of Spatial Relations between subgraphs expressed as formulae in the logic. Then we prove spatial entailments between properties of subgraphs using the axiomatization. In Section 4, we show that, given a subgraph at a detailed level, the logic allows to approximate it at a coarser level. Connection and other spatial relations between these approximated regions are also expressed. Finally we provide conclusions and further works.

2 The Logic UBiSKt

2.1 Kripke Semantics for UBiSKt

Let Prop be a countable set of propositional variables. Our syntax $\mathcal{L}$ for bi-intuitionistic stable tense logic with universal modalities consists of all logical connectives of bi-intuitionistic logic, i.e., two constant symbols $\bot$ and $\top$, disjunction $\vee$, conjunction $\wedge$, implication $\to$, coimplication $\prec$, and a finite set $\{\blacklozenge, \Box, \mathsf{A}, \mathsf{E}\}$ of modal operators. The set $\mathsf{Form}_{\mathcal{L}}$ of all formulas in $\mathcal{L}$ is defined inductively as follows:

$$\varphi ::= \top \mid \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \varphi \prec \varphi \mid \blacklozenge\varphi \mid \Box\varphi \mid \mathsf{E}\varphi \mid \mathsf{A}\varphi \quad (p \in \mathsf{Prop}).$$

We define the following abbreviations:

$$\neg\varphi := \varphi \to \bot, \quad {\lrcorner}\,\varphi := \top \prec \varphi, \quad \varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi),$$
$$\Diamond\,\varphi := {\lrcorner}\,\Box\neg\varphi, \quad \blacksquare\,\varphi := \neg\,\blacklozenge\,{\lrcorner}\,\varphi.$$

Definition 1 ([19]). *Let H be a preorder on a set U. We say that $X \subseteq U$ is an H-*set *if X is closed under H-successors, i.e., uHv and $u \in X$ jointly imply $v \in X$ for all elements u, $v \in U$. Given a preorder (U, H), a binary relation $R \subseteq U \times U$ is* stable *if it satisfies $H; R; H \subseteq R$.*

It is easy to see that a relation R on U is stable, if and only if, $R; H \subseteq R$ and $H; R \subseteq R$. Given any binary relation R on U, $\breve{R}$ is defined as the converse of R in the usual sense. Even if R is a stable relation on U, its converse $\breve{R}$ may be not stable.

Definition 2 ([19]). *The* left converse $\circlearrowleft R$ *of a stable relation R is $H; \breve{R}; H$.*

Definition 3. *We say that $F = (U, H, R)$ is an H-*frame *if U is a nonempty set, H is a preorder on U, and R is a* stable *binary relation on U. A* valuation *on an H-frame $F = (U, H, R)$ is a mapping V from Prop to the set of all H-sets on U. $M = (F, V)$ is an H-*model *if $F = (U, H, R)$ is an H-frame and V is a valuation. Given an H-model $M = (U, H, R, V)$, a state $u \in U$ and a formula φ, the satisfaction relation $M, u \models \varphi$ is defined inductively as follows:*

$$\begin{array}{lcl}
M, u \models p & \iff & u \in V(p), \\
M, u \models \top, & & \\
M, u \not\models \bot, & & \\
M, u \models \varphi \vee \psi & \iff & M, u \models \varphi \text{ or } M, u \models \psi, \\
M, u \models \varphi \wedge \psi & \iff & M, u \models \varphi \text{ and } M, u \models \psi, \\
M, u \models \varphi \to \psi & \iff & \text{For all } v \in U\ ((uHv \text{ and } M, v \models \varphi) \text{ imply } M, v \models \psi), \\
M, u \models \varphi \prec \psi & \iff & \text{For some } v \in U\ (vHu \text{ and } M, v \models \varphi \text{ and } M, v \not\models \psi), \\
M, u \models \blacklozenge\,\varphi & \iff & \text{For some } v \in U\ (vRu \text{ and } M, v \models \varphi), \\
M, u \models \Box\varphi & \iff & \text{For all } v \in U\ (uRv \text{ implies } M, v \models \varphi), \\
M, u \models \mathsf{E}\,\varphi & \iff & \text{For some } v \in U\ (M, v \models \varphi), \\
M, u \models \mathsf{A}\,\varphi & \iff & \text{For all } v \in U\ (M, v \models \varphi).
\end{array}$$

The truth set $[\![\varphi]\!]_M$ *of a formula φ in an H-model M is defined by $[\![\varphi]\!]_M := \{\, u \in U \mid M, u \models \varphi \,\}$. If the underlying model M in $[\![\varphi]\!]_M$ is clear from the context, we drop the subscript and simply write $[\![\varphi]\!]$. We write $M \models \varphi$ (read: 'φ is valid in M') to mean that $[\![\varphi]\!]_M = U$ or $M, u \models \varphi$ for all states $u \in U$. For a set Γ of formulas, $M \models \Gamma$ means that $M \models \gamma$ for all $\gamma \in \Gamma$. Given any H-frame $F = (U, H, R)$, we say that a formula φ is* valid *in F (written: $F \models \varphi$) if $(F, V) \models \varphi$ for any valuation V and any state $u \in U$, i.e., $[\![\varphi]\!]_{(F,V)} = U$.*

As for the abbreviated symbols, we may derive the following satisfaction conditions:

$$\begin{array}{lcl}
M, u \models \neg\varphi & \iff & \text{For all } v \in U\ (uHv \text{ implies } M, v \not\models \varphi), \\
M, u \models {\lrcorner}\,\varphi & \iff & \text{For some } v \in U\ (vHu \text{ and } M, v \not\models \varphi), \\
M, u \models \Diamond\,\varphi & \iff & \text{For some } v \in U\ ((v, u) \in \circlearrowleft R \text{ and } M, v \models \varphi), \\
M, u \models \blacksquare\,\varphi & \iff & \text{For all } v \in U\ ((u, v) \in \circlearrowleft R \text{ implies } M, v \models \varphi).
\end{array}$$

Proposition 1. *Given any H-model M, the truth set $[\![\varphi]\!]_M$ is an H-set.*

Proof. By induction on φ. When φ is of the form $\mathsf{E}\,\psi$, $\mathsf{A}\,\psi$, we remark that $[\![\varphi]\!]_M = U$ or $\varnothing$, which are both trivially H-sets. □

Definition 4. *Given a set $\Gamma \cup \{\varphi\}$ of formulas, φ is a* semantic consequence *of Γ (notation: $\Gamma \models \varphi$) if, whenever $M, u \models \gamma$ for all $\gamma \in \Gamma$, $M, u \models \varphi$ holds, for all H-models $M = (U, H, R, V)$ and all states $u \in U$. When Γ is a singleton $\{\psi\}$ of formulas, we simply write $\psi \models \varphi$ instead of $\{\psi\} \models \varphi$. When both $\varphi \models \psi$ and $\psi \models \varphi$ hold, we use $\varphi \dashv\models \psi$ to mean that they are equivalent with each other. When Γ is empty, we also simply write $\models \varphi$ instead of $\varnothing \models \varphi$.*

Table 1. Hilbert System H**UBiSKt**

Axioms and Rules for Intuitionistic Logic			
(A0)	$p \to (q \to p)$		
(A1)	$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$		
(A2)	$p \to (p \vee q)$	(A3)	$q \to (p \vee q)$
(A4)	$(p \to r) \to ((q \to r) \to (p \vee q \to r))$	(A5)	$(p \wedge q) \to p$
(A6)	$(p \wedge q) \to q$	(A7)	$(p \to (q \to p \wedge q))$
(A8)	$\bot \to p$	(A9)	$p \to \top$
(MP)	From φ and $\varphi \to \psi$, infer ψ		
(US)	From φ, infer a substitution instance φ' of φ		
Additional Axioms and Rules for Bi-intuitionistic Logic			
(A10)	$p \to (q \vee (p \prec q))$	(A11)	$((q \vee r) \prec q) \to r$
(Mon-$\prec$)	From $\delta_1 \to \delta_2$, infer $(\delta_1 \prec \psi) \to (\delta_2 \prec \psi)$		
Additional Axioms and Rules for Tense Operators			
(A12)	$p \to \Box\blacklozenge p$	(A13)	$\blacklozenge\Box p \to p$
(Mon$\Box$)	From $\varphi \to \psi$, infer $\Box\varphi \to \Box\psi$	(Mon$\blacklozenge$)	From $\varphi \to \psi$, infer $\blacklozenge\varphi \to \blacklozenge\psi$
Additional Axioms and Rules for Universal Modalities			
(A14)	$p \to \mathsf{A}\,\mathsf{E}\,p$	(A15)	$\mathsf{E}\,\mathsf{A}\,p \to p$
(A16)	$\mathsf{A}\,p \to p$	(A17)	$\mathsf{A}\,p \to \mathsf{A}\,\mathsf{A}\,p$
(A18)	$\mathsf{A}\,\neg p \leftrightarrow \neg\,\mathsf{E}\,p$	(A19)	$(\mathsf{A}\,p \wedge \mathsf{E}\,q) \to \mathsf{E}(p \wedge q)$
(A20)	$\mathsf{A}\,p \to \Box p$	(A21)	$(\mathsf{A}\,p \wedge \blacklozenge q) \to \blacklozenge(p \wedge q)$
(A22)	$(\mathsf{A}\,p \wedge (q \prec r)) \to ((p \wedge q) \prec r)$		
(Mon A)	From $\varphi \to \psi$, infer $\mathsf{A}\,\varphi \to \mathsf{A}\,\psi$	(Mon E)	From $\varphi \to \psi$, infer $\mathsf{E}\,\varphi \to \mathsf{E}\,\psi$

2.2 Hilbert System of Bi-intuitionistic Stable Tense Logic with Universal Modalities

Table 1 provides the Hilbert system H**UBiSKt**. Roughly speaking, it is a bi-intuitionistic tense analogue of a Hilbert system for the ordinary modal logic with the universal modalities [10] (see also [1, p.417]). In what follows in this paper, we assume that the reader is familiar with theorems and derived inference rules in intuitionistic logic. We define the notion of *theoremhood* in H**UBiSKt** as usual and write $\vdash_{\mathsf{HUBiSKt}} \varphi$ to mean that φ is a theorem of H**UBiSKt**. We say that φ is *provable* from Γ (notation: $\Gamma \vdash_{\mathsf{HUBiSKt}} \varphi$) if there is a finite set $\Gamma' \subseteq \Gamma$ such that $\vdash_{\mathsf{HUBiSKt}} \bigwedge \Gamma' \to \varphi$, where $\bigwedge \Gamma'$ is the conjunction of all elements of Γ' and $\bigwedge \Gamma' := \top$ when Γ' is an emptyset. When no confusion arises, we often simply write $\vdash \varphi$ and $\Gamma \vdash \varphi$ instead of $\vdash_{\mathsf{HUBiSKt}} \varphi$ and $\Gamma \vdash_{\mathsf{HUBiSKt}} \varphi$, respectively.

Theorem 1 (Soundness). *Given any formula φ, $\vdash_{\mathsf{HUBiSKt}} \varphi$ implies $\models \varphi$.*

Proof. Since H**UBiSKt** *without* universal modalities are already shown to be sound in [16], we focus on some of the new axioms and rules. Let $M = (U, H, R, V)$ be an H-model. Validity of axioms (A19), (A20) and (A22) are shown by the fact that $M, x \models p$ implies $[\![\mathsf{A}\, p]\!]_M = U$ for every $x \in U$. Let us check the validity of (A18) in detail. To show $\models \mathsf{A}\,\neg p \leftrightarrow \neg\,\mathsf{E}\, p$, it suffices to show $\mathsf{A}\,\neg p \dashv\models \neg\,\mathsf{E}\, p$. Fix any $x \in U$. Assume that $M, x \models \mathsf{A}\,\neg p$, which implies $[\![\neg p]\!]_M = U$. To show $M, x \models \neg\,\mathsf{E}\, p$, fix any $y \in U$ such that xHy. Our goal is to show $M, y \not\models \mathsf{E}\, p$, i.e., $V(p) = \varnothing$. But this is an easy consequence from $[\![\neg p]\!]_M = U$. Conversely, assume that $M, x \models \neg\,\mathsf{E}\, p$. Then $M, x \not\models \mathsf{E}\, p$ by xHx. This implies $V(p) = \varnothing$. To show $M, x \models \mathsf{A}\,\neg p$, fix any $y \in U$. Our goal is to establish $M, y \models \neg p$. But this is easy from $V(p) = \varnothing$. □

Our proof of the following proposition can be found in Appendix A.

Proposition 2. *All the following hold for* H**UBiSKt**.

1. $\vdash (\psi \prec \gamma) \to \rho$ *iff* $\vdash \psi \to (\gamma \vee \rho)$.
2. *If* $\vdash \varphi \leftrightarrow \psi$ *then* $\vdash (\gamma \prec \varphi) \leftrightarrow (\gamma \prec \psi)$.
3. $\vdash \neg(\varphi \prec \varphi)$.
4. $\vdash \varphi \vee \lrcorner\, \varphi$.
5. $\vdash \lrcorner\,\lrcorner\, \varphi \to \varphi$.
6. $\vdash \neg\varphi \to \lrcorner\, \varphi$.
7. $\vdash \varphi \to \neg\psi$ *iff* $\vdash \psi \to \neg\varphi$.
8. $\vdash \lrcorner\, \varphi \to \psi$ *iff* $\vdash \lrcorner\, \psi \to \varphi$.
9. $\vdash \lrcorner\,\neg\varphi \to \psi$ *iff* $\vdash \varphi \to \neg\,\lrcorner\, \psi$.
10. $\vdash \varphi \to \lrcorner\,\neg\varphi$.
11. $\vdash \neg\,\lrcorner\, \varphi \to \varphi$.
12. *If* $\vdash \varphi \to \psi$ *then* $\vdash \lrcorner\, \psi \to \lrcorner\, \varphi$.
13. $\vdash \lrcorner(\varphi \wedge \lrcorner\, \varphi)$.
14. $\vdash \mathsf{E}\,\varphi \to \psi$ *iff* $\vdash \varphi \to \mathsf{A}\,\psi$.
15. $\vdash \varphi \to \mathsf{E}\,\varphi$.
16. $\vdash \mathsf{E}\,\mathsf{E}\,\varphi \to \mathsf{E}\,\varphi$.
17. $\vdash \mathsf{A}\,\mathsf{E}\,\varphi \leftrightarrow \mathsf{E}\,\varphi$.
18. $\vdash \neg\,\mathsf{A}\,\varphi \leftrightarrow \lrcorner\,\mathsf{A}\,\varphi$.
19. $\vdash \mathsf{A}\,\varphi \vee \neg\,\mathsf{A}\,\varphi$.
20. $\vdash \neg\,\mathsf{E}\,\varphi \leftrightarrow \lrcorner\,\mathsf{E}\,\varphi$.
21. $\vdash \mathsf{E}\,\varphi \vee \neg\,\mathsf{E}\,\varphi$.
22. $\vdash \mathsf{E}\,\varphi \leftrightarrow \neg\,\mathsf{A}\,\neg\varphi$.
23. $\vdash \mathsf{A}(\lrcorner\, \varphi \to \psi) \leftrightarrow \mathsf{A}(\lrcorner\, \psi \to \varphi)$.
24. $\vdash \mathsf{E}(\lrcorner\,\neg\varphi \wedge \psi) \leftrightarrow \mathsf{E}(\varphi \wedge \lrcorner\,\neg\psi)$.

Theorem 2 (Strong Completeness of H**UBiSKt).** *If* $\Gamma \models \varphi$ *then* $\Gamma \vdash_{\mathsf{HUBiSKt}} \varphi$*, for every set* $\Gamma \cup \{\varphi\}$ *of formulas.*

Proof. We sketch our proof. First, let us introduce the terminology needed in this proof. A pair (Γ, Δ) of formulas is *provable* if $\vdash \bigwedge \Gamma' \to \bigvee \Delta'$ for some finite $\Gamma' \subseteq \Gamma$ and finite $\Delta' \subseteq \Delta$. A pair (Γ, Δ) is *complete* if $\Gamma \cup \Delta = \mathsf{Form}_{\mathcal{L}}$, i.e., $\varphi \in \Gamma$ or $\varphi \in \Delta$ for all formulas φ. Given a complete and unprovable pair (Γ, Δ), the canonical H-model $M_{(\Gamma,\Delta)} = (U, H, R, V)$ is defined as follows:

– $U := \{\, (\Sigma, \Theta) \mid (\Sigma, \Theta)$ is a complete unprovable pair and $(\Gamma, \Delta) S (\Sigma, \Theta) \,\}$ where the relation S is defined as: $(\Gamma, \Delta) S (\Sigma, \Theta) \iff (\mathsf{A}\,\varphi \in \Gamma$ iff $\mathsf{A}\,\varphi \in \Sigma)$ for all formulas φ.

– $(\Sigma_1, \Theta_1) H (\Sigma_2, \Theta_2)$ iff $\Sigma_1 \subseteq \Sigma_2$.

– $(\Sigma_1, \Theta_1) R (\Sigma_2, \Theta_2)$ iff $(\Box\varphi \in \Sigma_1$ implies $\varphi \in \Sigma_2)$ for all formulas φ.

– $(\Sigma, \Theta) \in V(p)$ iff $p \in \Sigma$.

We note that the relation S above is shown to be an equivalence relation and so U can be regarded as an S-equivalence class of the pair (Γ, Δ). This restriction makes the universal modalities A and E behave as “real” universal modalities in our canonical model. We also note that the canonical H-model $M_{(\Gamma,\Delta)}$ is shown to be a H-model by the same argument as in [16]. Then we can prove the following lemma (*Truth Lemma*): for any formula φ and any complete unprovable pair (Σ, Θ), we have the equivalence of $\varphi \in \Sigma \iff M_{(\Gamma,\Delta)}, (\Sigma, \Theta) \models \varphi$.

Based on these preparations, we can now prove strong completeness as follows. Fix any set $\Gamma \cup \{\, \varphi \,\}$ of formulas. We prove the contrapositive implication and so assume that $\Gamma \not\vdash \varphi$. It follows that $(\Gamma, \{\, \varphi \,\})$ is unprovable in H**UBiSKt**. By Lindenbaum construction, we can find a complete and Λ-unprovable pair $(\Sigma,\Theta) \in U$ such that $\Gamma \subseteq \Sigma$ and $\varphi \in \Theta$. By Truth Lemma, $M_{(\Sigma,\Theta)},(\Sigma,\Theta) \models \gamma$ for all $\gamma \in \Gamma$ and $M_{(\Sigma,\Theta)},(\Sigma,\Theta) \not\models \varphi$. Therefore, we conclude $\Gamma \not\models \varphi$. □

Theorem 3 (Decidability of HUBiSKt). *For every non-theorem φ of* **HUBiSKt***, there is a finite frame F such that $F \not\models \varphi$. Therefore,* **HUBiSKt** *is decidable.*

Proof. We focus on the former part, since the latter part follows from the former part and the finite axiomatization of **HUBiSKt**. We employ an appropriate filtration technique here. We sketch our proof. Suppose that φ is not a theorem of **HUBiSKt**. By Theorem 2, there is an H-model M such that $M \not\models \varphi$. Put Δ as the set of all subformulas of φ. Define a finite H-model $M^s_\Delta = (U_\Delta, \underline{H}^+_\Delta, \underline{R}^s_\Delta, V_\Delta)$ by:

- $U_\Delta = \{\, [x] \mid x \in U \,\}$ and $[x]$ is an equivalence class of the relation $\sim_\Delta$, which is defined by:

$$x \sim_\Delta y \iff (M, x \models \varphi \text{ iff } M, y \models \varphi) \text{ for all } \varphi \in \Delta.$$

- $\underline{H}^+_\Delta$ is the transitive closure of the relation $\underline{H}_\Delta$ which is defined by

$$[x]\underline{H}_\Delta[y] \iff x'Hy' \text{ for some } x' \in [x] \text{ and some } y' \in [y],$$

- $\underline{R}^s_\Delta$ is defined as $\underline{H}^+_\Delta; \underline{R}_\Delta; \underline{H}^+_\Delta$ where the relation $\underline{R}_\Delta$ is defined as follows:

$$[x]\underline{R}_\Delta[y] \iff x'Ry' \text{ for some } x' \in [x] \text{ and some } y' \in [y].$$

- $V_\Delta(p) = \{\, [x] \mid x \in V(p) \,\}$ for all $p \in \Delta$.

We note that U_Δ is finite since Δ is the set of all subformulas of φ. By induction on $\psi \in \Delta$, we can show that the equivalence $M, x \models \psi \iff M_\Delta, [x] \models \psi$ holds for every $x \in U$ and every $\psi \in \Delta$. Then it follows from $M \not\models \varphi$ that $M^s_\Delta \not\models \varphi$ hence $F^s_\Delta \not\models \varphi$, where F^s_Δ is the frame part of M^s_Δ, as desired. □

2.3 Tableau-System for UBiSKt.

Tab**UBiSKt** is a tableau-system for **UBiSKt**. It has been also implemented using the theorem-prover generator *MetTel* [21]. Our implementation of Tab**UBiSKt** is available at [17]. We are going to show that Tab**UBiSKt** is equipollent with **HUBiSKt**, and so the tableau with its implementation can be seen as a computational tool for reasoning with **UBiSKt**.

Expressions in the calculus have one of these forms:

$$s : S\varphi \qquad \bot \qquad sHt \qquad sRt \qquad s \approx t \qquad s \not\approx t$$

where S denotes a sign, either T for true or F for false, and s, t are names or labels from a fixed set Label in the tableau language whose intended meaning are elements of U.

Let Tab**UBiSKt** be the extension of Tab**BiSKt**, as described in [19] plus the following rules (for the full tableau calculus, see Table 3 in Appendix C):

$$\frac{s : T(\mathsf{A}\,\varphi), \quad t : S\psi}{t : T\varphi}\,(T\,\mathsf{A}) \qquad \frac{s : F(\mathsf{A}\,\varphi)}{m : F\varphi}\,(F\,\mathsf{A})\ m \text{ is fresh in the branch}$$

$$\frac{s : T(\mathsf{E}\,\varphi)}{m : T\varphi}\,(T\,\mathsf{E})\ m \text{ is fresh in the branch} \qquad \frac{s : F(\mathsf{E}\,\varphi), \quad t : S\psi}{t : F\varphi}\,(F\,\mathsf{E})$$

As in ordinary tableau calculi, rules in Tab**UBiSKt** are used to decompose formulae analyzing their main connective. Since some rules are branching or splitting, the tableau derivation process constructs a tree. If a branch in the tableau derivation ends with $\bot$, then the branch is said to be *closed*. If a branch is not closed, then it is *open*. If a branch is open and no more rules can be applied to it then the branch is *fully-expanded*. A tableau is closed when all its branches are closed, it is open otherwise. The derivation process stops when all the branches in the tableau derivation are either closed or fully expanded. An open fully expanded branch will give the information for building model for a set of tableau expressions given as derivation input. A formula φ is a *theorem* in Tab**UBiSKt** if a tableau derivation for the input set $\{a : F\varphi\}$, where 'a' is a constant label which is intended to represent the initial world, will give a closed tableau. A formula φ is provable from a finite set Γ of formulae if a tableau derivation for the input set $\{a : T\Gamma\} \cup \{a : F\varphi\}$ will give a closed tableau, where $a : T\Gamma$ means $(a : T\gamma)$, for all $\gamma \in \Gamma$.

Proofs of the following two theorems are found in Appendices C and D.

Theorem 4 (Soundness of TabUBiSKt). *Given a finite set $\Gamma \cup \{\varphi\}$ of formulae, if φ is provable from Γ in* Tab**UBiSKt** *then $\Gamma \models \varphi$.*

Theorem 5. *Given a formula $\varphi \in \mathsf{Form}_{\mathcal{L}}$ the following are equivalent:*

1. *φ is a theorem in* H**UBiSKt***,*
2. *φ is a theorem in* Tab**UBiSKt***,*
3. *φ is valid in all H-models.*

Theorem 5 shows that the proof systems H**UBiSKt** and Tab**UBiSKt** capture the same set of theorems. Since H**UBiSKt** is decidable (Theorem 3), the tableau-system Tab**UBiSKt** can be seen as the specification of a concrete algorithm for deciding whether a formula $\varphi \in \mathsf{Form}_{\mathcal{L}}$ is a theorem in H**UBiSKt**.

3 Reasoning with Spatial Relations in UBiSKt

3.1 UBiSKt as a Logic for Graphs

The logic **UBiSKt** is an expansion of the logic **BiSKt**, introduced in [19] and also studied in [16]. As is already noted in [19], a special case of an H-model is where the set U is the set of all edges and nodes of a multigraph, and H is the incidence relation as follows.

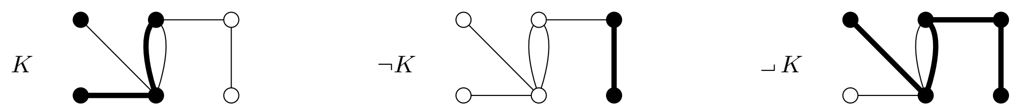


Fig. 2. The two kinds of complement of a subgraph K.

Definition 5. *A multigraph G consists of two disjoints sets E and N called the edges and the nodes, together a function $i : E \mapsto \mathcal{P}(N)$ such that for all $e \in E$ the cardinality of $i(e)$ is* *either* 1 *or* 2*, and where* $\mathcal{P}(N)$ *is the powerset of the set of nodes. Note that these are undirected multigraphs and that edges may be loops incident only with a single node. A subgraph* K *of* G *is a subset of* G *such that given* $u \in K$*, if* $v \in i(u)$ *then* $v \in K$*.*

In [11] multigraphs are also called pseudographs. Any multigraph gives rise to a pre-order from which the structure of edges and nodes can be re-captured. Let $G = (E, N, i)$ be a multigraph. Define $U = E \cup N$ and define a relation $H \subseteq U \times U$ by $(u, v) \in H$ if and only if either 1) u is an edge and $v \in i(u)$, or 2) $u = v$. It is clear that H is reflexive and transitive. A structure (U, H) obtained from a multigraph in this way, uniquely determines the original multigraph, as the nodes are those elements $u \in U$ such that for all $v \in U$, $(u, v) \in H$ implies $u = v$.

Fig. 3. The multi-graph on the left has four nodes, a, b, c, d, and four edges w, x, y, z. The corresponding pre-order for this multi-graph is the reflexive closure of the relation on the set $\{a, b, c, d, w, x, y, z\}$ shown on the right hand side.

Fig. 3 shows a multigraph and the associated pre-order. From now on we will refer to multigraphs simply as graphs.

It is easy to see that the subgraphs of a graph $G = (U, H)$ are exactly the subsets $K \subseteq U$ that are closed under H-successor. Therefore the notion of H-set as in Definition 1 corresponds to the notion of subgraph. Since any formula φ in the logic is interpreted on the H-set $[\![\varphi]\!]_M$, formulae in the logic can be regarded as names for subgraphs of an underlying graph $G = (U, H)$. Similarly, operations in the logic provides with operations on subgraphs, following the semantics defined in Section 2.1. Fig. 2 shows a graph with a subgraph and the two operations of complement $\neg$ and $\lrcorner$, where the leftmost is a graph G with subgraph K and the remaining graphs are the subgraphs obtained by the operation $\neg$ and $\lrcorner$. We note, in Fig. 2, that $\neg K$ is the largest subgraph disjoint from K and $\lrcorner K$ is the smallest subgraph whose union with K gives all of the underlying graph G.

In the next section we are going to use two negations and universal modalities in **UBiSKt** to encode spatial relations between subgraphs, where subgraphs are naturally thought of as discrete regions of the space, i.e., sets of single nodes and edges between them.

3.2 Topological Notions in UBiSKt

Definition 6. *Let* X *be a Heyting Algebra with bottom element* 0 *and top element* 1*, and let* $c : X \to X$ *be a function. We say that* (X, c) *is a* Čech closure algebra *if for all* x*,* $y \in X$*:*

$$c(0) = 0, \quad x \leq c(x), \quad c(x \vee y) = c(x) \vee c(y).$$

Given a function $i : X \to X$*, We say that* (X, i) *is a* Čech interior algebra *if for all* x*,* $y \in X$*:*

$$i(1) = 1, \qquad i(x) \leq x, \quad i(x \wedge y) = i(x) \wedge i(y).$$

Let M be an H-model. Since **UBiSKt** is an expansion of intuitionistic logic, it is easy to see that $\{ [\![\varphi]\!]_M \mid \varphi \in \mathsf{Form}_\mathcal{L} \}$ forms a Heyting algebra by interpreting $\bot$ as the bottom element 0 and $\top$ as the top element 1. Then, as we already noted in [18], $\lrcorner\neg$ enables us to define a Čech closure algebra. This can be also verified by our axiomatization. Since the adjunction "$\lrcorner\neg \dashv \neg\lrcorner$", the combination $\lrcorner\neg$ preserves finite disjunctions and the combination $\neg\lrcorner$ preserves finite conjunctions (due to item 9 of Proposition 2), we can easily obtain the first and the third conditions for a Čech closure algebra by soundness of **HUBiSKt**. Moreover the second condition follows from item 10 of Proposition 2 and soundness of **HUBiSKt**. Dually, we can also similarly verify that the combination $\neg\lrcorner$ gives rise to a Čech interior algebra on $\{ [\![\varphi]\!]_M \mid \varphi \in \mathsf{Form}_\mathcal{L} \}$.

We may regard $\lrcorner\neg$ and $\neg\lrcorner$ as $\lozenge$ and $\blacksquare$ arising from the left converse $\circlearrowleft H$ of H, respectively. This is explained as follows. When we restrict our attention to the class of H-models $M = (U, H, R, V)$ satisfying $R = H$, we note that the modal operators $\lozenge$ and $\blacksquare$ arising from the left converse $\circlearrowleft R$ of R are equivalent with $\lrcorner\neg$ and $\neg\lrcorner$, respectively, while the modal operators $\blacklozenge$ and $\Box$ become trivial in the sense that $\blacklozenge\varphi \leftrightarrow \varphi$ and $\Box\varphi \leftrightarrow \varphi$ are valid in the model.

Definition 7. *Given an H-model $M = (U, H, R, V)$, an H-set $K \subseteq U$ is* representable *in the syntax $\mathcal{L}$ of* **UBiSKt** *if there is a formula $\varphi \in \mathsf{Form}_\mathcal{L}$ such that $K = [\![\varphi]\!]_M$.*

With the help of two kinds of negations $\neg$ and $\lrcorner$, we can talk about the notions of boundary and exterior of a representable subgraph.

$\partial^N(\varphi) := \varphi \wedge \lrcorner\varphi$ represents the nodes-boundary of a subgraph $[\![\varphi]\!]_M$.

$\partial(\varphi) := \neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi$ represents the general boundary of a subgraph $[\![\varphi]\!]_M$. This is the node-boundary plus the edges in the subgraph between these nodes.

$\neg\varphi$ represents the exterior of the subgraph $[\![\varphi]\!]_M$.

Beside the topological notions of closure, interior and boundary of a subgraph, we also introduce the following formulae whose semantic meaning will often occur in the rest of the paper:

$\mathsf{E}\,\varphi$ represents $[\![\varphi]\!]_M \neq \varnothing$.
$\mathsf{A}\,\neg\varphi$ or $\neg\,\mathsf{E}\,\varphi$ represents $[\![\varphi]\!]_M = \varnothing$.
$\mathsf{A}\,\varphi$ represents $[\![\varphi]\!]_M = U$.
$\mathsf{E}\lrcorner\varphi$ represents $[\![\varphi]\!]_M \neq U$.

Using the closure operator the spatial relation of connection between subgraphs $[\![\varphi]\!]$ and $[\![\psi]\!]$ can be expressed by an appropriate formula in **UBiSKt**:

$$C(\varphi, \psi) := \mathsf{E}(\lrcorner\neg\varphi \wedge \psi).$$

The formula states that $[\![\lrcorner\neg\varphi]\!]_M \cap [\![\psi]\!]_M \neq \varnothing$. This means that the two subgraphs are connected if they are an edge apart, in the limit case. This notion of connection is the equivalent of the notion of adjacency found in Galton [8], and is one of the notions of connection expressed by closure in [5], with the difference the the operation $\lrcorner\neg$ is not a Kuratowski closure, but a Čech closure.

Beside connection the following Spatial Relations can be defined inside **UBiSKt**: Part, non Part, Proper Part, Non-tangential Proper Part, Tangential Proper Part, External Connection, Disconnection, Partial overlapping, Equality, and the Inverse of Non-tangential Proper Part and Tangential Proper Part respectively. We list each relation with its correspondent formula in Table 2. We note that classically the relation of $PP(\varphi, \psi)$ could be defined as $\mathsf{A}(\varphi \to \psi)$ and $\mathsf{E}(\psi \wedge \neg\varphi)$ that is equivalent to $\mathsf{E}(\psi \prec \varphi)$, where we note that $\psi \prec \varphi$ is defined as $\psi \wedge \neg\varphi$ in classical logic. However, this is not the case in our setting. Take the example of a graph with two nodes and an edge between them. Say a formula φ represents the two nodes and ψ represents the whole graph.

Then $PP(\varphi,\psi)$ holds. However, it is true that $\mathsf{A}(\varphi \to \psi)$, but it is not the case that $E(\psi \wedge \neg\varphi)$ since $[\![\neg\varphi]\!]$ is empty. Therefore, in order to cover also this kind of cases we use $\mathsf{E}(\psi \prec \varphi)$ that is not equivalent, intuitionistically, to $\mathsf{E}(\psi \wedge \neg\varphi)$, and expresses the right notion of *non-P*(ψ,φ), as can be checked from the semantic definition of $\prec$ in Section 2.1.

Table 2. Spatial Relations and the corresponding formulae

Spatial Relation	Formula	Spatial Relation	Formula
$P(\varphi,\psi)$	$\mathsf{A}(\varphi \to \psi)$	$DC(\varphi,\psi)$	$\mathsf{A}\,\neg(\lrcorner\neg\varphi \wedge \psi)$
$\textit{non-P}(\varphi,\psi)$	$\mathsf{E}(\varphi \prec \psi)$	$PO(\varphi,\psi)$	$\mathsf{E}(\varphi \wedge \psi) \wedge \textit{non-P}(\varphi,\psi) \wedge \textit{non-P}(\psi,\varphi)$
$PP(\varphi,\psi)$	$P(\varphi,\psi) \wedge \textit{non-P}(\psi,\varphi)$	$EQ(\varphi,\psi)$	$\mathsf{A}(\varphi \leftrightarrow \psi)$
$NTPP(\varphi,\psi)$	$PP(\varphi,\psi) \wedge P(\lrcorner\neg\varphi,\psi)$	$NTPP^i(\varphi,\psi)$	$NTPP(\psi,\varphi)$
$TPP(\varphi,\psi)$	$PP(\varphi,\psi) \wedge non{-}\mathrm{P}(\lrcorner\neg\varphi,\psi)$	$TPP^i(\varphi,\psi)$	$TPP(\psi,\varphi)$
$EC(\varphi,\psi)$	$C(\varphi,\psi) \wedge \mathsf{A}(\neg(\varphi \wedge \psi))$		

3.3 Reasoning on Spatial Entailments in UBiSKt

In this section we are going to show some interesting entailments between spatial properties of subgraphs, that can be derived syntactically in **UBiSKt**. Indeed all the following has been proved using H**UBiSKt**. For these axiomatic proofs the reader is referred to Appendix B. We remark that all of Propositions 3-18 are also mechanically verified in our implementation of Tab**UBiSKt** in terms of *MetTel* [21]. The implemented prover with instructions on how to use it and how to prove any of the following propositions can be found at [17].

Proposition 3. $\vdash_{\mathsf{HUBiSKt}} \mathsf{E}(\lrcorner\neg\varphi \wedge \psi) \leftrightarrow \mathsf{E}(\varphi \wedge \lrcorner\neg\psi)$. *If the closure of a region intersects another region, then the closure of this latter region will intersect the former.*

This holds due to item 24 of Proposition 2. From Proposition 3 we can infer that the spatial relation of Connection, $C(\varphi,\psi)$ can also be expressed by the formula $\mathsf{E}(\varphi \wedge \lrcorner\neg\psi)$, showing that our formulation is equivalent to the notion of connection C_2 found in [5].

Proposition 4. i) $\vdash_{\mathsf{HUBiSKt}} P(\lrcorner\neg\varphi,\psi) \leftrightarrow P(\varphi,\neg\lrcorner\psi)$. *If the closure of a region is part of another region then the former region will be part of the interior of the latter, and vice versa.*

ii) $\vdash_{\mathsf{HUBiSKt}} NTP(\neg\lrcorner\varphi,\varphi)$ *The interior of a region is always Non-tangential part of the region, where* $NTP(\varphi,\psi) := P(\varphi,\psi) \wedge P(\lrcorner\neg\varphi,\psi)$.

For i) of Proposition 4, if we add the extra assumption that the first region is also proper part of the second one, then we obtain an alternative definition of Non-tangential proper part, that makes use of the interior operator instead of the closure: $NTPP(\varphi,\psi) := \mathsf{A}(\varphi \to \psi) \wedge \mathsf{E}(\psi \prec \varphi) \wedge \mathsf{A}(\varphi \to \neg\lrcorner\psi)$. For ii) of Proposition 4, the result is restricted to the notion of Non-tangential Part and it does not extend to the notion of Non-tangential *Proper* Part because, as we shall see later, there are regions that are equal to their own interior.

Proposition 5. $\vdash_{\mathsf{HUBiSKt}} \partial^N(\varphi) \leftrightarrow \partial^N(\varphi) \wedge \lrcorner\,\partial^N(\varphi)$. *The Nodes-boundary of a region is always boundary of itself.*

We might want to distinguish between Boundary-regions, whose interior is empty and Substantial-regions, whose interior is non-empty interior.

Definition 8. $BR(\varphi) := \mathsf{E}(\varphi) \wedge EQ(\varphi, \partial(\varphi))$. *A region is a Boundary-Region in a model if it is not empty and it is equal to its own general-boundary.*

Proposition 6. i) $\vdash_{\mathbf{HUBiSKt}} BR(\varphi) \to P(\varphi, \neg(\neg \lrcorner\, \varphi))$. *If φ is a Boundary-Region then it is part of the exterior of its own interior.*

ii) $\vdash_{\mathbf{HUBiSKt}} P(\psi, \neg\delta) \to \neg\, \mathsf{E}(\psi \wedge \delta)$. *If a region is part of the exterior of another region, then the two regions do not overlap.*

iii) $\vdash_{\mathbf{HUBiSKt}} \mathsf{E}(\psi) \wedge \neg\, \mathsf{E}(\psi \wedge \neg \lrcorner\, \psi) \to \mathsf{A}\neg(\neg \lrcorner\, \psi)$. *If a region is non-empty and it does not overlap its own interior, then the interior of that region is empty.*

From item ii) of this proposition, we can obtain $\vdash_{UBiSKt} BR(\varphi) \to \neg\, \mathsf{E}(\varphi \wedge \neg \lrcorner\, \varphi)$, i.e., If a region is a Boundary region, then it does not overlap with its own interior.

From the results contained in Proposition 6 we can infer the following.

Proposition 7. $\vdash_{\mathbf{HUBiSKt}} BR(\varphi) \to \mathsf{A}\neg(\neg \lrcorner\, \varphi)$. *If a region is a boundary-region then its interior is empty.*

Proposition 8. $\vdash_{\mathbf{HUBiSKt}} \mathsf{A}\neg(\neg \lrcorner\, \varphi) \to EQ(\varphi, \partial(\varphi))$. *If the interior of a region is empty, then the region is equal to its own boundary. With the extra assumption that the region is non-empty we have that if its interior is empty then it is a Boundary-region.*

Propositions 7 and 8 show that our notion of Boundary-region is equivalent to the notion of non-empty region whose interior is empty:

Proposition 9. $\vdash_{\mathbf{HUBiSKt}} BR(\varphi) \leftrightarrow \mathsf{E}\,\varphi \wedge \mathsf{A}\neg(\neg \lrcorner\, \varphi)$

Definition 9. *A region is a Substantial-region if its interior is not empty:* $SR(\varphi) := \mathsf{E}\neg \lrcorner\, \varphi$.

Proposition 10. $\vdash_{\mathbf{HUBiSKt}} \mathsf{E}(\lrcorner\, \neg\varphi) \to \mathsf{E}(\varphi)$. *If the interior of a region is non-empty, then the region is non-empty.*

Proposition 11. $\vdash_{\mathbf{HUBiSKt}} SR(\varphi) \to$ *not-*$P(\varphi, \partial(\varphi))$. *If a region has non-empty interior then it is not part of its own boundary. Therefore it is not a Boundary-region.*

Proposition 12. $\nvdash_{\mathbf{HUBiSKt}} \mathsf{E}(\neg \lrcorner\, \varphi) \wedge \mathsf{E}(\lrcorner\, \varphi) \to \mathsf{E}\,\partial(\varphi)$. *If a region has non-empty interior and it is not the whole graph, it does not necessarily have a boundary.*

Proof. A counter-model is a graph made of two single nodes and no edges between them. The region represented by φ is a node, so that it is not the whole graph. Since the region is a node not connected to anything else, because the graph has no edges, its interior is the region itself and, for the same region, its boundary is empty. □

Thanks to the distinction between Boundary regions and Substantial Regions, we are able to refine the spatial relations given above, by limiting the domain of the regions considered. As an example, the relation of Partial overlapping PO can be defined only between regions with non-empty interior, and be distinguished into three different relations.

Let φ, ψ represent substantial regions:

$$PO_1(\varphi,\psi) := \mathsf{E}(\varphi \wedge \psi) \wedge \mathit{not}\text{-}P(\varphi,\psi) \wedge \mathit{not}\text{-}P(\psi,\varphi) \wedge \mathsf{E}((\neg\neg(\varphi \wedge \lrcorner\,\varphi) \wedge \varphi) \wedge (\neg\neg(\psi \wedge \lrcorner\,\psi) \wedge \psi)),$$

$$PO_2(\varphi,\psi) := \mathsf{E}(\varphi \wedge \psi)\mathit{not}\text{-}P(\varphi,\psi) \wedge \mathit{not}\text{-}P(\psi,\varphi) \wedge \mathsf{E}(\neg \lrcorner\,\varphi \wedge \neg \lrcorner\,\psi) \wedge BR(\neg \lrcorner\,\varphi \wedge \neg \lrcorner\,\psi),$$

$$PO_3(\varphi,\psi) := \mathsf{E}(\varphi \wedge \psi)\mathit{not}\text{-}P(\varphi,\psi) \wedge \mathit{not}\text{-}P(\psi,\varphi) \wedge \mathsf{E}(\neg \lrcorner\,\varphi \wedge \neg \lrcorner\,\psi) \wedge SR(\neg \lrcorner\,\varphi \wedge \neg \lrcorner\,\psi).$$

where $PO_1(\varphi,\psi)$ expresses the idea that the two regions overlap in their boundaries, $PO_2(\varphi,\psi)$ expresses that two regions overlap in their interior and region where they overlap is a Boundary-region, $PO_3(\varphi,\psi)$ expresses that two regions overlap in their interior, and the region where they overlap is in turn a Substantial-region.

Proposition 13. $\vdash_{\mathbf{HUBiSKt}} \neg SR(\partial^N(\varphi) \wedge \partial^N(\psi))$ *The intersection of the Node-boundaries of two regions is not a Substantial-Region.*

Proposition 14. $\vdash_{\mathbf{HUBiSKt}} EQ(\neg\partial^N(\varphi), \neg\partial(\varphi))$. *The exterior of the Node-Boundary is equal to the exterior of the general-Boundary.*

Proposition 15. i) $\vdash_{\mathbf{HUBiSKt}} SR(\varphi) \wedge \mathsf{A}\,\neg\partial^N(\varphi) \rightarrow \mathsf{E}\,\varphi \wedge P(\varphi, \neg \lrcorner\,\varphi)$. *If a region is a Substantial region and its Node-Boundary is empty, then region is part of its own interior.*

ii) $\vdash_{\mathbf{HUBiSKt}} P(\varphi, \neg \lrcorner\,\varphi) \wedge \mathsf{E}\,\varphi \rightarrow SR(\varphi) \wedge \mathsf{A}\,\neg\partial^N(\varphi)$. *If a region is part of its own interior and it is non-empty then it is a Substantial-region and its boundary is empty.*

By the results in Proposition 14, we can generalize the results obtained in Propositions 15 to the general-Boundary of a region.

Proposition 16. i) $\vdash_{\mathbf{HUBiSKt}} SR(\varphi) \wedge \mathsf{A}\,\neg\partial(\varphi) \rightarrow \mathsf{E}\,\varphi \wedge P(\varphi, \neg \lrcorner\,\varphi)$.

ii) $\vdash_{\mathbf{HUBiSKt}} P(\varphi, \neg \lrcorner\,\varphi) \wedge \mathsf{E}\,\varphi \rightarrow SR(\varphi) \wedge \mathsf{A}\,\neg\partial(\varphi)$.

Proposition 17. $\vdash_{\mathbf{HUBiSKt}} P(\psi, \neg \lrcorner\,\psi) \rightarrow P(\neg \lrcorner\,\psi, \psi)$. *If a region is part of its own interior, then the interior is part of the region. Therefore a region is part of its interior if and only if it is equal to if own interior:* $\vdash_{\mathbf{HUBiSKt}} P(\psi, \neg \lrcorner\,\psi) \leftrightarrow EQ(\psi, \neg \lrcorner\,\psi)$.

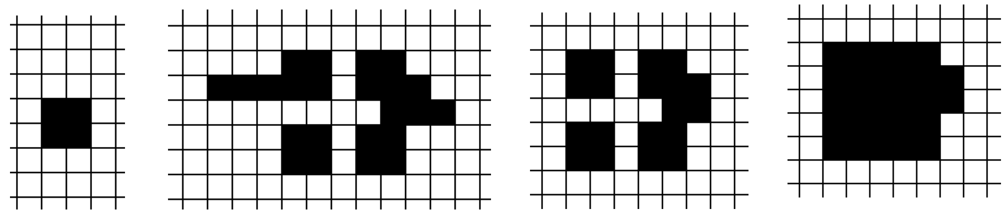
From results in Propositions 16 and 17 we can infer the following.

Proposition 18. $\vdash_{\mathbf{HUBiSKt}} \mathsf{E}\,\varphi \wedge EQ(\varphi, \neg \lrcorner\,\varphi) \leftrightarrow SR(\varphi) \wedge \mathsf{A}\,\neg\partial(\varphi)$. *A non-empty region is equal to its own interior if and only if it is a substantial region and it has empty boundary.*

Definition 10. *Given a region representable by a formula φ, we define the notion of isolated component of the underling graph-space as follows: $IC(\varphi) := \mathsf{A}(\varphi \leftrightarrow \neg \lrcorner\,\varphi) \wedge \mathsf{E}\,\varphi$. We say that a formula φ represents an isolated component of the underlying graph-space if it is equal to its own interior. By the additional assumption that $\mathsf{E}\,\varphi$ holds we exclude the empty region as a possible isolated component of the graph.*

4 Granular Spatial Relations

The idea of zooming out, or viewing a situation in a less detailed way, is commonplace. Intuitively, zooming out on an image (a set of pixels) we expect narrow cracks to fuse and narrow spikes to become invisible. This intuitive expectation is bourne out in the formalisation due to mathematical morphology. The idea here is that instead of being able to see individual pixels, only groups of pixels can be seen. This is illustrated in Fig. 4 using the operations of opening and closing by a structuring element. For details of mathematical morphology see [12], but here it is sufficient to know that the opening consists of the image formed by (overlapping) copies of the structuring element within the original, and that closing consists of the complement of the (overlapping) copies of the structuring element but rotated by half a turn, that can be placed wholly outside the original.



Structuring element | original image | opening of original image | closing of opening of original image

Fig. 4. Approximation of a subset of $\mathbb{Z}^2$ by a 2×2 structuring element.

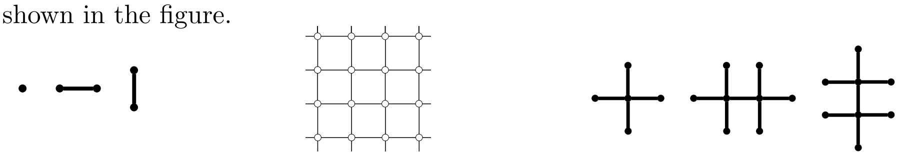
As explained in [12] the operations of mathematical morphology are not restricted to approximating subsets of a grid of pixels by a structuring element, but apply in the context of any subset of set with an arbitrary binary relation on the set instead of a structuring element. As [19] shows, we can extend this to a pre-order (U, H) and approximate H-sets in this structure by means of a stable relation R. Given $X \subseteq U$, we use $X \oplus R$ (dilation of X) to denote $\{u \in U \mid \exists v(v\ R\ u \wedge v \in X)\}$, and use $R \ominus X$ (erosion of X) to denote $\{u \in U \mid \forall v(u\ R\ v \Rightarrow v \in X)\}$. It is well known that for R fixed the operations $\_\oplus R$ and $R \ominus \_$ form an adjunction from the lattice $\mathcal{P}(U)$ to itself, with $\_\oplus R$ left adjoint to $R \ominus \_$. From adjunction, some properties of dilation and erosion follow, for example, given two sets A and B and a relation R, $A \oplus R \subseteq B$ is equivalent to $A \subseteq R \ominus B$. The opening of X by R is denoted $X \circ R$ and defined as $(R \ominus X) \oplus R$ and the closing is $X \bullet R = R \ominus (X \oplus R)$. The connection between mathematical morphology and modal logic has been studied in [2] in the set based case, and extended to the graph based case in [19]. Here, the modalities $\lozenge$, $\blacklozenge$, $\square$ and $\blacksquare$ function as semantic operators taking H-sets to H-sets, with $\lozenge$ associated to $X \mapsto X \oplus \curvearrowright R$, $\blacklozenge$ associated to $X \mapsto X \oplus R$, $\square$ associated to $R \ominus X$ and $\blacksquare$ associated to $\curvearrowright R \ominus X$. So, given a propositional variable p representing an H-set, opening and closing of the H-set are expressible in the logic by the formulae $\blacklozenge \square p$ and $\square \blacklozenge p$ respectively. This extends to **UBiSKt**, as it is an extension of the logic studied in [19]. In this setting, the idea of opening as fitting copies of a structuring element inside an image remains meaningful. Copies of the structuring element correspond to R-dilates in the following sense.

Definition 11. *A subset $X \subseteq U$ is an R-dilate if $X = \{u\} \oplus R$ for some $u \in U$.*

Stability implies that R-dilates are always H-sets. It is straightforward to check that opening and closing can be expressed in terms of dilates:

$$X \circ R = \bigcup \{\{u\} \oplus R \mid \{u\} \oplus R \subseteq X\}, \qquad X \bullet R = \{u \in U \mid \{u\} \oplus R \subseteq \bigcup \{\{x\} \oplus R \mid x \in X\}\}.$$

To give concrete examples, let (U, H) be the graph with $\mathbb{Z}^2$ for nodes and two nodes are connected by an edge if exactly one of their coordinates differs by 1. We refer to this as the graph $\mathbb{Z}^2$, visualized as in Fig 5. The dilates by H and by $\curvearrowright H$ of a node, a horizontal edge, and a vertical edge are shown in the figure.



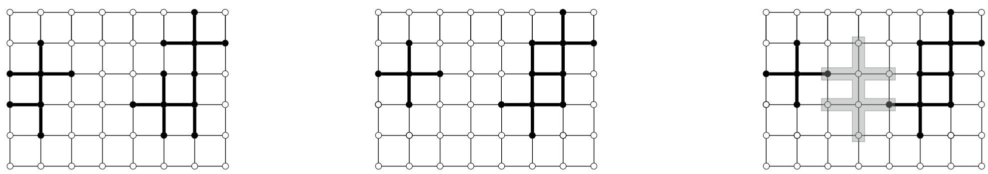
Dilates of H $\quad$ (U, H) is infinite grid, $\mathbb{Z}^2$ $\quad$ Dilates of $\curvearrowright H$

Fig. 5. Shapes of the dilates of H and of $\curvearrowright H$ when (U, H) is the graph shown.

We can think of $(X \circ R) \bullet R$ as a granular version of X in which we cannot 'see' arbitrary H-sets, but only ones that can be described in terms of the R-dilates. As we have seen, opening and closing correspond to specific sequences of modalities in the logic. So, given a representable H-set, we can capture its granular version by a formula in the logic.

Definition 12. *Given a propositional variable p representing an H-set, 'coarsely p' is defined by $\mathbb{G}p := \Box\blacklozenge\blacklozenge\Box p$.*

We notice that the closing of the opening of a region is known in mathematical morphology as an alternating filter. This gives a way of zooming-out for a region, but how should we define connection between coarse regions? The issue is that the space underlying the regions should become coarser – regions disconnected may become connected for example. In the same way that coarse regions are described in terms of dilates, a coarse version of connection can be formulated using dilates. To motivate this consider Fig 6 which shows the idea that coarse regions are coarsely connected if there is a dilate intersecting both, or visually and informally that the gap between can be bridged by a dilate. Requiring an R-dilate joining two regions seems a suitable notion of coarse connection, as it extends the intuition of connection at the detailed level. Indeed two H-sets X and Y are connected at the detailed level (see table 2 for the formula) if the gap between them can be bridged by an H-dilate, so if they are an edge apart, in the limit case. Going to the granular level, single H-dilates are no longer "visible", and the space has coarser atomic parts: copies of the structuring element, i.e. R-dilates.



Two H-sets $\quad$ Their approximations by opening then closing by $\curvearrowright H$ $\quad$ The approximations can be joined by a dilate

Fig. 6. Granular View by Relation $\curvearrowright H$

Definition 13. *An R-dilate, D, joins H-sets X and Y if $X \cap D \neq \varnothing$ and $Y \cap D \neq \varnothing$.*

It is easy to see that asking for an R-dilates that joins X and Y amounts to ask that, given the union of the R-dilates intersecting X, at least one that intersects Y.

Lemma 1 ([19]). *If R and S are relations on a set U and $X \subseteq U$ then $X \oplus (R; S) = (X \oplus R) \oplus S$.*

Lemma 2. *Let X be an H-set and R a stable relation. The union of the R-dilates intersecting X is $X \oplus (\circlearrowleft R; R)$.*

Proof. First we show that the union of the R-dilate intersecting X is $X \oplus \breve{R} \oplus R$. If $\{u\} \oplus R$ intersects X, for some $u \in U$, then there is a $x \in X$ such that $\{u\} \subseteq \{x\} \oplus \breve{R}$. Hence $\{u\} \oplus R \subseteq \{x\} \oplus \breve{R} \oplus R \subseteq X \oplus \breve{R} \oplus R$. In the other direction, if $y \in X \oplus \breve{R} \oplus R$, then there is some $u \in U$ and $x \in X$ such that uRy and uRx, so that $y \in \{u\} \oplus R$ with $\{u\} \oplus R$ intersecting X. Now, since $\breve{R} \subseteq \circlearrowleft R$ (see definition 2), $X \oplus \breve{R} \oplus R \subseteq X \oplus \circlearrowleft R \oplus R = X \oplus \circlearrowleft R; R$. Also $X \oplus \circlearrowleft R; R = X \oplus H; \breve{R}; H; R = X \oplus \breve{R}; H; R \subseteq X \oplus \breve{R}; R = X \oplus \breve{R} \oplus R$ because X is an H-set and R is stable. So $X \oplus \breve{R} \oplus R = X \oplus \circlearrowleft R; R$.

Proposition 19. *There is an R-dilate joining H-sets X and Y iff $(X \oplus (\circlearrowleft R\,; R)) \cap Y \neq \varnothing$.*

Proof. The union of R-dilates intersecting X is $X \oplus (\circlearrowleft R; R)$ from lemma 2. This intersect Y iff $(X \oplus (\circlearrowleft R; R)) \cap Y \neq \varnothing$.

The above discussion provides a semantic justification for the following definition.

Definition 14. *Coarse connection is defined by $C_{\mathbb{G}}(p, q) := \mathsf{E}(\blacklozenge \Diamond \mathbb{G}p \wedge \mathbb{G}q)$.*

Note that when $R = H$, then $\mathbb{G}p$ is equivalent to p and $C_{\mathbb{G}}(p, q)$ is equivalent to $C(p, q)$. Indeed, as noticed in section 3.2, $\lrcorner\neg$ can be regarded as $\Diamond$ arising from the left converse of H, $\circlearrowleft H$, and $\blacklozenge\varphi \leftrightarrow \varphi$ and $\Box\varphi \leftrightarrow \varphi$ are valid in a model where $R = H$. Another special case is when H is the identity relation on a set, and R is an equivalence relation. In this case $[\![\mathbb{G}p]\!]$ will correspond to the lower approximation, in the sense of rough-set theory, of $[\![p]\!]$.

As we would expect, our notion of coarse connection is symmetric.

Proposition 20. $\vdash_{\mathbf{HUBiSKt}} \mathsf{E}(\blacklozenge \Diamond\varphi \wedge \psi) \leftrightarrow \mathsf{E}(\varphi \wedge \blacklozenge \Diamond\psi)$.

Proof. See Appendix B

Similar to connection, we can define a notion of coarse parthood in terms of R-dilates. The standard notion of parthood at the detailed level (Table 2) says that, given H-sets X and Y, X is part of Y if and only if all the atomic H-dilates in X lie in Y. A suitable notion of coarse parthood will require that X is coarse part of Y if and only if all the R-dilates in X lie also in Y.

Proposition 21. *Let X and Y be H-sets, and R a stable relation. The following are equivalent: 1) all the R-dilates in X lie in Y and 2) $R \ominus (X) \subseteq R \ominus (Y)$.*

Proof. The union of all the R-dilates in X is the opening of X: $X \circ R = (R \ominus X) \oplus R$. Hence, requiring the all the R-dilates in X lie in Y amounts to require that $(R \ominus X) \oplus R \subseteq Y$. By properties of adjunction this is equivalent to $R \ominus X \subseteq R \ominus Y$.

Lemma 3 ([19]). *Let φ, ψ be formulae in **UBiSKt** with $[\![\varphi]\!]_M$ and $[\![\psi]\!]_M$ associated H-sets and let M be and H-model. Then $[\![\varphi]\!]_M \subseteq [\![\psi]\!]_M$ iff $M \models \mathsf{A}(\varphi \to \psi)$*

The above reasoning together with lemma 3 provide a semantic justification for the following definition of coarse parthood between coarse regions.

Definition 15. *Coarse parthood is defined by $P_G(p, q) := \mathsf{A}(\Box\mathbb{G}p \to \Box\mathbb{G}q)$.*

The negation of the notion of coarse parthood will give a notion of coarse non-parthood: this requires that there is at least an R-dilate in X such that this is not in Y. From proposition 21, we know that this is equivalent to $R \ominus X \not\subseteq R \ominus Y$.

Lemma 4. *Let φ, ψ be formulae in **UBiSKt** with $[\![\varphi]\!]_M$ and $[\![\psi]\!]_M$ associated H-sets and let M be and H-model. Then $[\![\varphi]\!]_M \not\subseteq [\![\psi]\!]_M$ iff $M \models \mathsf{E}(\varphi \prec \psi)$.*

Proof. See Appendix E

Because of lemma 4 we have the following definition.

Definition 16. *Coarse non-parthood is defined by non-$P_G(p, q) := \mathsf{E}(\Box\mathbb{G}p \prec \Box\mathbb{G}q)$.*

We now analyze how to extend the spatial relation of overlapping to the granular level. Two H-sets X and Y overlaps at the detailed level if and only if there is at least a non-empty H-dilate that lies both in X and Y. Following this idea, a suitable notion of coarse overlapping requires a non-empty R-dilate that lies both in X and Y.

Proposition 22. *Let X and Y be H-sets and R a stable relation. The following are equivalent:* 1) *there is a non empty R-dilate that lies both in X and in Y and* 2) $(X \cap Y) \circ R \neq \varnothing$.

Proof. $(X \cap Y) \circ R$ is the opening of $X \cap Y$, so union of all R-dilates both in X and in Y. Hence requiring that there is a non empty R-dilate that lies both in X and in Y amounts to require that the opening of $X \cap Y$ is not empty: $(X \cap Y) \circ R \neq \varnothing$.

Hence coarse overlapping between coarse regions can be defined as follows:

Definition 17. *Coarse overlapping is defined by $O_G(p, q) := \mathsf{E}(\blacklozenge\Box(\mathbb{G}p \wedge \mathbb{G}q))$.*

As an example, in Fig. 7 (left) we show two H-sets (red and black) such that they intersect, but an R-dilate will not fit inside the region of intersection ($R = \smile H$). Therefore the spatial relation O_G will not hold. If the region of intersection is at least as big as an R-dilate (right), then the relation O_G will hold.

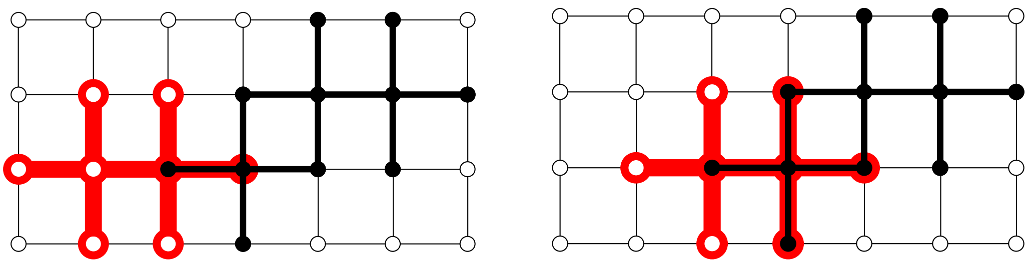
Given H-sets X and Y, X is non-tangential part of Y at the detailed level if and only if X is part of Y and the closure of X, $\lrcorner\neg X$, is still part of Y. This means that all the H-dilates that intersect X lie in Y. Similarly, a suitable notion of coarse non-tangential part between H-sets X and Y can be formalized by requiring that X is coarse part of Y and all the R-dilates intersecting X lie in Y.

Proposition 23. *Let X and Y be H-sets and R a stable relation. The following are equivalent: 1) all the R-dilates overlapping X lie in Y, and 2) $X \oplus \circlearrowleft R \subseteq R \ominus Y$.*

Proof. The union of the R-dilates overlapping X lie in Y is $(X \oplus \circlearrowleft R \oplus R) \subseteq Y$ by Lemma 2. This is equivalent to $X \oplus \circlearrowleft R \subseteq R \ominus Y$ by properties of adjunction.

The above reasoning provides a semantic justification for the following definition.

Definition 18. *Coarse non-tangential part is defined by $NTP_G(p,q) := \mathsf{A}(\Box\mathbb{G}p \to \Box\mathbb{G}q) \wedge \mathsf{A}(\Diamond\mathbb{G}p \to \Box\mathbb{G}q)$.*



Two H-sets not sharing a whole R-dilate

Two H-sets sharing a whole R-dilate

Fig. 7. Cases of coarse non-overlapping and of coarse overlapping, where R is $\circlearrowleft H$.

Finally, we analyze the notion of coarse tangential part. At the detailed level, an H-set X is tangential part of Y if it is its part and there is at least an H-dilate intersecting X that does not lie in Y. This is obtained by requiring that the closure of X is not part of Y. Hence, at the granular level we will require that the union of all R-dilates intersecting X does not lie in Y. This mean that we have to negate the requirement for NTP_G: by proposition 23 this is $X \oplus \circlearrowleft R \nsubseteq R \ominus Y$. By this and Lemma 4 we have the following.

Definition 19. *Coarse tangential part is defined by $TP_G(p,q) := \mathsf{A}(\Box\mathbb{G}p \to \Box\mathbb{G}q) \wedge \mathsf{E}(\Diamond\mathbb{G}p \prec \Box\mathbb{G}q)$.*

5 Conclusions and Further Work

We have provided a sound and complete axiomatisation for the logic **UBiSKT** and used this to prove a number of results in Section 3.3 demonstrating that the definitions of discrete spatial relations have properties appropriate to the spatial concepts involved. We have also provided a tableau calculus for the logic, and proved that this is equivalent to the Hilbert-style axiomatisation. While spatial relations in discrete space have been studied before, the novelty in our work here is the use of reasoning in a formal logic together with an implementation of a theorem-proving procedure for the logic.

There are several directions for further work. Our use of **UBiSKT** to formulate a notion of coarsening fits in with existing work observing that both rough set theory and mathematical morphology are closely connected with modal logic [3]. Our definitions of coarse spatial relations in this setting are, however, a novelty. We have been able to indicate the semantic basis and some basic properties of these relations. In future work we will investigate the use of the axiomatisation in establishing more general forms of connection. For example, by measuring the connection between two regions at two levels of detail, that is the value of $(C(p,q), C_{\mathbb{G}}(p,q))$, we anticipate based on the evidence in [20] (which considered granularity but not the connection relation) that spatial relations able to make finer distinctions can be obtained.

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A Proof of Proposition 2

Proof. We provide proofs of some of them. Our proof for item 2 proceed as follows: Suppose that $\vdash \varphi \leftrightarrow \psi$. We prove that $\vdash (\gamma \prec \varphi) \to (\gamma \prec \psi)$ alone, since the other direction is similarly shown. By the axiom (A10), we have $\vdash \gamma \to (\psi \vee (\gamma \prec \psi))$. By the supposition, we obtain $\vdash \gamma \to (\varphi \vee (\gamma \prec \psi))$, which implies $\vdash (\gamma \prec \varphi) \to (\gamma \prec \psi))$, as desired. Item 3 is shown as follows. Since $\vdash \varphi \to (\varphi \vee \bot)$ is a theorem of intuitionistic logic, we obtain $\vdash (\varphi \prec \varphi) \to \bot$ by item 1. For item 4, we show $\vdash \varphi \vee (\top \prec \varphi)$, which is equivalent to $\vdash \top \to (\varphi \vee (\top \prec \varphi))$. We can derive this from $\vdash (\top \prec \varphi) \to (\top \prec \varphi)$ by item 1. Item 5 is shown by items 4 and 1. Item 6 can be derived with the help of item 4. Items 7 and 8 are easy. Item 9 is an easy consequence of items 7 and 8. Item 10 is shown as follows. Since $\vdash \varphi \to \neg\neg\varphi$ and $\vdash \neg\neg\varphi \to \lrcorner\neg\varphi$ by item 6, we obtain $\vdash \varphi \to \lrcorner\neg\varphi$, as desired. Item 11 is similarly shown to item 10. Item 12 is shown as: Suppose that $\vdash \varphi \to \psi$. By item 5, we obtain $\vdash \lrcorner\lrcorner\varphi \to \psi$. By item 8, we conclude $\vdash \lrcorner\psi \to \lrcorner\varphi$, as desired. Let us move to item 13. We proceed as follows: By intuitionistic logic we have $\vdash (\varphi \wedge \lrcorner\varphi) \to \varphi$. Then we deduce from item 12 that $\vdash \lrcorner\varphi \to \lrcorner(\varphi \wedge \lrcorner\varphi)$, which implies $\vdash (\varphi \wedge \lrcorner\varphi) \to \lrcorner(\varphi \wedge \lrcorner\varphi)$ by intuitionistic logic. By item 5, we get $\vdash \lrcorner\lrcorner(\varphi \wedge \lrcorner\varphi) \to \lrcorner(\varphi \wedge \lrcorner\varphi)$. With the help of item 1 and the definition of $\lrcorner$, this is equivalent with $\vdash \lrcorner(\varphi \wedge \lrcorner\varphi)$, as required.

Items 15 and 16 are established by (A16) $\mathsf{A}\,p \to p$ and (A17) $\mathsf{A}\,p \to \mathsf{A}\,\mathsf{A}\,p$ by item 14 respectively. For item 17, it suffices to show the left-to-right implication since the other half is just an instance of (A16). The left-to-right direction is obtained from item 16 by item 14. For item 18, it suffices to prove $\vdash \lrcorner\mathsf{A}\,\varphi \to \neg\mathsf{A}\,\varphi$ since the other half is shown by item 6. To show our goal, it suffices to show $\vdash ((\top \prec \mathsf{A}\,\varphi) \wedge \mathsf{A}\,\varphi) \to \bot$. By (A22), we obtain $\vdash ((\top \prec \mathsf{A}\,\varphi) \wedge \mathsf{A}\,\varphi) \to ((\mathsf{A}\,\varphi \wedge \top) \prec \mathsf{A}\,\varphi)$. By noting that $((\mathsf{A}\,\varphi \wedge \top) \prec \mathsf{A}\,\varphi) \to \bot$ by items 3 and 2, we can obtain our goal. Item 19 is established from items 4 and 19. For item 20, the left-to-right implication is just item 6 and the other half is shown as follows. It suffices to show $\vdash (\lrcorner\mathsf{E}\,\varphi \wedge \mathsf{E}\,\varphi) \to \bot$. By item 17, we show $\vdash ((\top \prec \mathsf{E}\,\varphi) \wedge \mathsf{A}\,\mathsf{E}\,\varphi) \to \bot$. Since $\vdash ((\top \prec \mathsf{E}\,\varphi) \wedge \mathsf{A}\,\mathsf{E}\,\varphi) \to ((\top \wedge \mathsf{E}\,\varphi) \prec \mathsf{E}\,\varphi)$ by (A22) and $\vdash ((\top \wedge \mathsf{E}\,\varphi) \prec \mathsf{E}\,\varphi) \to \bot$ by items 2 and 3. Item 21 is established similarly to item 19. For item 22, we proceed as follows. For the left-to-right direction, we suffices to show that $\vdash (\mathsf{E}\,\varphi \wedge \mathsf{A}\,\neg\varphi) \to \bot$. But this is easy from the axiom (A19) and $\vdash \mathsf{E}\,\bot \leftrightarrow \bot$. For the right-to-left direction, it suffices to prove $\vdash (\neg\mathsf{A}\,\neg\varphi \wedge \neg\mathsf{E}\,\varphi) \to \bot$ by item 21. By the axiom (A18), we show $\vdash (\neg\mathsf{A}\,\neg\varphi \wedge \mathsf{A}\,\neg\varphi) \to \bot$, which is trivial. Let us move to item 23. It suffices to show the left-to-right implication. To show it, it suffices to prove $\vdash \mathsf{A}(\lrcorner\varphi \to \psi) \to (\lrcorner\psi \to \varphi)$ by the rule (MonA) and the axiom (A17). By the axiom (A22), we get $\vdash (\mathsf{A}(\lrcorner\varphi \to \psi) \wedge \lrcorner\psi) \to ((\lrcorner\varphi \to \psi) \prec \psi)$. We note that $\vdash (\lrcorner\varphi \to \psi) \to (\psi \vee \varphi)$, which is derivable with the help of $\vdash \varphi \vee \lrcorner\varphi$ (due to item 4). It follows from item 1 that $\vdash ((\lrcorner\varphi \to \psi) \prec \psi) \to \varphi$. Thus, we get $\vdash (\mathsf{A}(\lrcorner\varphi \to \psi) \wedge \lrcorner\psi) \to \varphi$, which is equivalent with $\vdash (\mathsf{A}(\lrcorner\varphi \to \psi) \to (\lrcorner\psi \to \varphi)$. This finishes to prove item 23. Finally, let us show item 24. By item 23, we obtain $\vdash \mathsf{A}(\lrcorner\neg\varphi \to \neg\psi) \leftrightarrow \mathsf{A}(\lrcorner\neg\psi \to \neg\varphi)$, which is equivalent with $\vdash \mathsf{A}\,\neg(\lrcorner\neg\varphi \wedge \psi) \leftrightarrow \mathsf{A}\,\neg(\lrcorner\neg\psi \wedge \varphi)$. This implies $\vdash \neg\mathsf{A}\,\neg(\lrcorner\neg\varphi \wedge \psi) \leftrightarrow \neg\mathsf{A}\,\neg(\lrcorner\neg\psi \wedge \varphi)$. By item 22 of $\vdash \mathsf{E}\,\theta \leftrightarrow \neg\mathsf{A}\,\neg\theta$, we obtain our desired goal. □

B Axiomatic Proofs of Propositions 4-18 in Section 3.3 and Proposition 20 in Section 4

Proposition 4: i) it follows from item 18 of Proposition 2. ii) Since $NTP(\neg\lrcorner\varphi, \varphi) := \mathsf{A}(\neg\lrcorner\varphi \to \varphi) \wedge \mathsf{A}(\lrcorner\neg\neg\lrcorner\varphi \to \varphi)$, we proceed as follows.

1. $\vdash \lrcorner\neg\neg\lrcorner\varphi \to \lrcorner\lrcorner\neg\lrcorner\varphi$, because in general $\vdash \neg\neg\alpha \to \lrcorner\neg\alpha$ from item 5 of Proposition 2 and take $\alpha = \lrcorner\varphi$.

2. $\vdash\ \lrcorner\lrcorner\neg\lrcorner\varphi \to \neg\lrcorner\varphi$, because $\lrcorner\lrcorner\alpha \to \alpha$ (from item 7 of Proposition 2) and take $\neg\lrcorner\varphi$ as α.

3. $\vdash \neg\lrcorner\varphi \to \varphi$ by item 10 of Proposition 2.

4. $\vdash (\lrcorner\neg\neg\lrcorner\varphi \to \varphi)$ by concatenation of lines 1, 2 and 3.

5. $\vdash \mathsf{A}(\lrcorner\neg\neg\lrcorner\varphi \to \varphi)$ by necessitation rule.

Proposition 5: $\partial^N(\varphi) \leftrightarrow \partial^N(\varphi) \wedge \lrcorner\partial^N(\varphi) := (\varphi \wedge \lrcorner\varphi) \leftrightarrow (\varphi \wedge \lrcorner\varphi) \wedge \lrcorner(\varphi \wedge \lrcorner\varphi)$.

1. $\vdash \lrcorner(\varphi \wedge \lrcorner\varphi)$ by item 13 of Proposition 2.

2. $\vdash \lrcorner(\varphi \wedge \lrcorner\varphi) \to ((\varphi \wedge \lrcorner\varphi) \to \lrcorner(\varphi \wedge \lrcorner\varphi))$ by axiom `A0`.

3. $\vdash (\varphi \wedge \lrcorner\varphi) \to \lrcorner(\varphi \wedge \lrcorner\varphi)$ by `MP`.

4. $\vdash (\varphi \wedge \lrcorner\varphi) \to (\varphi \wedge \lrcorner\varphi) \wedge \lrcorner(\varphi \wedge \lrcorner\varphi)$.

5. $\vdash (\varphi \wedge \lrcorner\varphi) \wedge \lrcorner(\varphi \wedge \lrcorner\varphi) \to (\varphi \wedge \lrcorner\varphi)$ follows as an instance of `A5`

6. $\vdash (\varphi \wedge \lrcorner\varphi) \leftrightarrow (\varphi \wedge \lrcorner\varphi) \wedge \lrcorner(\varphi \wedge \lrcorner\varphi)$ by lines 4 and 5.

Proposition 6:

i) $BR(\varphi) \to P(\varphi, \neg\neg\lrcorner\varphi) := \mathsf{E}\,\varphi \wedge \mathsf{A}(\varphi \leftrightarrow (\neg\neg(\varphi \wedge \lrcorner\varphi))) \to \mathsf{A}(\varphi \to \neg\neg\lrcorner\varphi)$

1. $\vdash \mathsf{E}\,\varphi \wedge \mathsf{A}(\varphi \leftrightarrow \neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi) \to (\varphi \leftrightarrow \neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi)$ by `A16`.

2. $\vdash (\varphi \leftrightarrow \neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi) \to (\varphi \to \neg\neg(\varphi \wedge \lrcorner\varphi))$

3. $\vdash (\varphi \to \neg\neg(\varphi \wedge \lrcorner\varphi)) \to (\varphi \to \neg\neg\varphi \wedge \neg\neg\lrcorner\varphi)$, because $\neg\neg(\alpha \wedge \beta) \leftrightarrow \neg\neg\alpha \wedge \neg\neg\beta$ is a theorem in intuitionistic logic.

4. $\vdash (\varphi \to \neg\neg\varphi \wedge \neg\neg\lrcorner\varphi) \to (\varphi \to \neg\neg\lrcorner\varphi)$

5. $\vdash \mathsf{E}\,\varphi \wedge \mathsf{A}(\varphi \leftrightarrow \neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi) \to (\varphi \to \neg\neg\lrcorner\varphi)$ by concatenating 1-4.

6. $\vdash \mathsf{A}(\mathsf{E}\,\varphi \wedge \mathsf{A}(\varphi \leftrightarrow \neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi)) \to \mathsf{A}(\varphi \to \neg\neg\lrcorner\varphi)$ by `Mon` A rule.

So we have shown $\mathsf{A}(BR(\varphi)) \to P(\varphi, \neg\neg\lrcorner\varphi)$. But:

7. $\vdash \mathsf{A}(\mathsf{E}\,\varphi \wedge \mathsf{A}(\varphi \leftrightarrow (\neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi))) \leftrightarrow (\mathsf{A}\,\mathsf{E}\,\varphi \wedge \mathsf{A}\,\mathsf{A}(\varphi \leftrightarrow (\neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi)))$ because $\vdash \mathsf{A}(\alpha \wedge \beta) \leftrightarrow \mathsf{A}(\alpha) \wedge \mathsf{A}(\beta)$ due to the adjunction "$\mathsf{E} \dashv \mathsf{A}$" by item 14 of Proposition 2.

8. $\vdash (\mathsf{A}\,\mathsf{E}\,\varphi \wedge \mathsf{A}\,\mathsf{A}(\varphi \leftrightarrow (\neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi))) \leftrightarrow \mathsf{E}\,\varphi \wedge \mathsf{A}(\varphi \leftrightarrow (\neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi))$ by $\vdash \mathsf{A}\,\mathsf{A}\,\alpha \leftrightarrow \mathsf{A}\,\alpha$ (due to `A16` and `A17`) and $\vdash \mathsf{A}\,\mathsf{E}\,\alpha \leftrightarrow \mathsf{E}\,\alpha$ (due to item 17 of Proposition 2).

9. $\vdash \mathsf{E}\,\varphi \wedge \mathsf{A}(\varphi \leftrightarrow \neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi) \to \mathsf{A}(\varphi \to \neg\neg\lrcorner\varphi)$ by lines 6, 7 and 8.

ii) $\mathsf{A}(\psi \to \neg\delta) \to \neg\,\mathsf{E}(\psi \wedge \delta)$.

1. $\vdash \mathsf{A}(\psi \to \neg\delta) \to \mathsf{A}\,\neg(\psi \wedge \delta)$ because $(\alpha \to \neg\beta) \leftrightarrow \neg(\alpha \wedge \beta)$ is a theorem of intuitionistic logic.

2. $\vdash \mathsf{A}\,\neg(\psi \wedge \delta) \to \neg\,\mathsf{E}(\psi \wedge \delta)$ by `A18`.

3. $\vdash \mathsf{A}(\psi \to \neg\delta) \to \neg\,\mathsf{E}(\psi \wedge \delta)$ by concatenating lines 1 and 2.

iii) $\neg\,\mathsf{E}(\varphi \wedge \neg\lrcorner\varphi) \to \mathsf{A}\,\neg\neg\lrcorner\varphi$.

1. $\vdash \neg\lrcorner\varphi \to (\varphi \wedge \neg\lrcorner\varphi)$ by item 11 of Proposition 2.

2. $\vdash \neg(\varphi \wedge \neg \lrcorner \varphi) \to \neg\neg \lrcorner \varphi$ by intuitionistic logic.

3. $\vdash \mathsf{A}\neg(\varphi \wedge \neg \lrcorner \varphi) \to \mathsf{A}\neg\neg \lrcorner \varphi$ by `Mon A`.

4. $\vdash \neg \mathsf{E}(\varphi \wedge \neg \lrcorner \varphi) \to \mathsf{A}\neg\neg \lrcorner \varphi$. by `A18`.

Proposition 8: $\mathsf{A}\neg(\neg \lrcorner \varphi) \to EQ(\varphi, \partial(\varphi)) := \mathsf{A}\neg(\neg \lrcorner \varphi) \to \mathsf{A}(\varphi \leftrightarrow (\neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi))$.

1. $\vdash \neg\neg \lrcorner \varphi \to (\varphi \to \neg\neg \lrcorner \varphi)$ by `A0`.

2. $\vdash \mathsf{A}(\neg\neg \lrcorner \varphi) \to \mathsf{A}(\varphi \to \neg\neg \lrcorner \varphi)$ by `Mon A`.

3. $\vdash \mathsf{A}(\neg\neg \lrcorner \varphi) \to \mathsf{A}(\varphi \to \neg\neg\varphi)$ because $\alpha \to \neg\neg\alpha$ is a theorem following from item 5 of Proposition 2

4. $\vdash \mathsf{A}(\neg\neg \lrcorner \varphi) \to \mathsf{A}(\varphi \to \neg\neg \lrcorner \varphi) \wedge \mathsf{A}(\varphi \to \neg\neg\varphi)$ from lines 3 and 4.

5. $\vdash \mathsf{A}(\neg\neg \lrcorner \varphi) \to \mathsf{A}(\varphi \to \neg\neg\varphi \wedge \neg\neg \lrcorner \varphi)$ because A preserves conjunction.

6. $\vdash \mathsf{A}(\neg\neg \lrcorner \varphi) \to \mathsf{A}(\varphi \to \neg\neg(\varphi \wedge \lrcorner \varphi))$, because $\neg\neg(\alpha \wedge \beta) \leftrightarrow \neg\neg\alpha \wedge \neg\neg\beta$ is a theorem in intuitionistic logic.

7. $\vdash \mathsf{A}(\neg\neg \lrcorner \varphi) \to \mathsf{A}(\varphi \to \neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi)$

8.$\vdash \mathsf{A}(\neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi \to \varphi)$

9. $\vdash \mathsf{A}(\neg\neg \lrcorner \varphi) \to \mathsf{A}(\varphi \leftrightarrow \neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi)$ from 8 and 9.

Proposition 10: $\mathsf{E}(\neg \lrcorner \varphi) \to \mathsf{E}\varphi$:

1. $\vdash (\neg \lrcorner \varphi) \to \varphi$ item 10 of Proposition 2.

2. $\vdash \mathsf{E}(\neg \lrcorner \varphi) \to \mathsf{E}\varphi$ by Mon E rule.

Proposition 11: $SR(\varphi) \to$ not-$P(\varphi, \partial(\varphi)) := \mathsf{E}(\neg \lrcorner \varphi) \to \mathsf{E}(\varphi \prec (\neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi))$.

First we show that

$\vdash (\alpha \wedge \neg\beta) \to (\alpha \prec \beta)$:

1. $\vdash \alpha \to (\beta \vee (\alpha \prec \beta))$ by axiom `A10`.

2. $\vdash (\alpha \wedge \neg\beta) \to (\alpha \prec \beta)$ by intuitionistic logic.

By putting

$\alpha := \varphi, \quad \beta := \neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi)$

we obtain:

3. $\vdash (\varphi \wedge \neg(\neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi)) \to (\varphi \prec \neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi)$

4. $\vdash \neg \lrcorner \varphi \to (\varphi \wedge \neg(\neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi))$.

5. $\vdash \neg \lrcorner \varphi \to \varphi$ by item 11 of Proposition 2.

6. $\vdash \neg \lrcorner \varphi \wedge \lrcorner \varphi \to \bot$

7. $\vdash \neg \lrcorner \varphi \wedge (\varphi \wedge \lrcorner \varphi) \wedge \varphi \to \bot$

8. $\vdash \neg \lrcorner \varphi \wedge \neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi \to \bot$

9. $\vdash \neg \lrcorner \varphi \to \neg(\neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi)$

10. $\vdash \neg \lrcorner \varphi \to (\varphi \wedge \neg(\neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi))$ by lines 5 and 9

11. $\vdash \neg \lrcorner \varphi \to (\varphi \prec \neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi)$ by concatenating lines 10 and 3.

12. $\vdash \mathsf{E}(\neg \lrcorner \varphi) \to \mathsf{E}((\varphi \prec \neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi))$ by `Mon-E` rule

Proposition 13: $\neg SR(\partial^N \varphi \wedge \partial^N(\psi)) := \neg\, \mathsf{E} \neg \lrcorner((\varphi \wedge \lrcorner \varphi) \wedge (\psi \wedge \lrcorner \psi))$

1. $\vdash \lrcorner(\psi \wedge \lrcorner \psi)$ from item 13 of Proposition 2.

2. $\vdash \neg\neg \lrcorner(\psi \wedge \lrcorner \psi)$ by intuitionistic logic.

3. $\vdash \mathsf{A} \neg\neg \lrcorner(\psi \wedge \lrcorner \psi)$ by necessitation rule.

4. $\vdash ((\psi \wedge \lrcorner \psi) \wedge (\varphi \wedge \lrcorner \varphi)) \to (\psi \wedge \lrcorner \psi)$ by `A6`

5. $\vdash \lrcorner(\psi \wedge \lrcorner \psi) \to \lrcorner((\varphi \wedge \lrcorner \varphi) \wedge (\psi \wedge \lrcorner \psi))$ by item 12 of Proposition 2

6. $\vdash \mathsf{A} \neg\neg \lrcorner(\psi \wedge \lrcorner \psi) \to \mathsf{A} \neg\neg \lrcorner((\varphi \wedge \lrcorner \varphi) \wedge (\psi \wedge \lrcorner \psi))$ by intuitionistic logic and `Mon A`

7. $\vdash \mathsf{A} \neg\neg \lrcorner((\varphi \wedge \lrcorner \varphi) \wedge (\psi \wedge \lrcorner \psi))$ by concatenation of lines 3 and 6.

Proposition 14: $EQ(\neg\partial^N(\varphi), \partial^N(\psi) := \mathsf{A}((\neg(\varphi \wedge \lrcorner \varphi)) \leftrightarrow (\neg((\neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi))))$

1. $\vdash (\neg(\varphi \wedge \lrcorner \varphi) \wedge (\neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi)) \to \bot$ because $\neg\alpha \wedge \neg\neg\alpha \to \bot$ that is is a theorem in intuitionistic logic.

2. $\vdash \neg(\neg(\varphi \wedge \lrcorner \varphi) \wedge (\neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi))$ from line 1.

3. $\vdash \neg(\varphi \wedge \lrcorner \varphi) \to (\neg(\neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi))$ because $(\alpha \wedge \beta) \to (\alpha \wedge \neg\beta)$ is a theorem in intuitionistic logic.

4. $\vdash (\varphi \wedge \lrcorner \varphi) \to (\neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi)$ because $\vdash \alpha \to \neg\neg\alpha$.

5. $\vdash (\neg(\neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi)) \to \neg(\varphi \wedge \lrcorner \varphi)$ by line 4.

6. $\vdash \neg(\varphi \wedge \lrcorner \varphi) \leftrightarrow (\neg(\neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi))$ by lines 3 and 5.

7. $\vdash \mathsf{A}(\neg(\varphi \wedge \lrcorner \varphi) \leftrightarrow (\neg(\neg\neg(\varphi \wedge \lrcorner \varphi) \wedge \varphi)))$ by necessitation rule.

Proposition 15:

i) $SR(\varphi) \wedge \mathsf{A} \neg\partial^N(\varphi) \to \mathsf{E}\, \varphi \wedge P(\varphi, \neg \lrcorner \varphi) := \mathsf{E} \neg \lrcorner \varphi \wedge \mathsf{A} \neg(\varphi \wedge \lrcorner \varphi) \to \mathsf{E}\, \varphi \wedge \mathsf{A}(\varphi \to \neg \lrcorner \varphi)$.

1. $\vdash \mathsf{E} \neg \lrcorner \varphi \to \mathsf{E}\, \varphi$ by Proposition 10.

2. $\vdash \mathsf{A} \neg(\varphi \wedge \lrcorner \varphi) \to \mathsf{A}(\varphi \to \neg \lrcorner \varphi)$ because $\neg(\alpha \wedge \beta) \to (\alpha \to \neg\beta)$ is a theorem in intuitionistic logic.

3. $\vdash \mathsf{E} \neg \lrcorner \varphi \wedge \mathsf{A} \neg(\varphi \wedge \lrcorner \varphi) \to \mathsf{E}\, \varphi \wedge \mathsf{A}(\varphi \to \neg \lrcorner \varphi)$. from 1 and 2.

ii) $P(\varphi, \neg \lrcorner \varphi) \wedge \mathsf{E}\, \varphi \to SR(\varphi) \wedge \mathsf{A} \neg\partial^N(\varphi) := \mathsf{A}(\varphi \to \neg \lrcorner \varphi) \wedge \mathsf{E}\, \varphi \to \mathsf{E}(\neg \lrcorner \varphi) \wedge \mathsf{A} \neg(\varphi \wedge \lrcorner \varphi)$.

1. $\vdash \mathsf{A}(\varphi \to \neg \lrcorner \varphi) \wedge \mathsf{E}\, \varphi \to \mathsf{E}((\varphi \to \neg \lrcorner \varphi) \wedge \varphi) \to \mathsf{E}(\neg \lrcorner \varphi)$ by `A19` and `MP`.

2. $\vdash \mathsf{A}(\varphi \to \neg \lrcorner \varphi) \wedge \mathsf{E}\, \varphi \to \mathsf{E}(\neg \lrcorner \varphi)$ from 1.

3. $\vdash (\mathsf{A}(\varphi \to \neg \lrcorner \varphi) \wedge \mathsf{E}\, \varphi) \to (\mathsf{A} \neg(\varphi \wedge \lrcorner \varphi) \wedge (\mathsf{E} \neg \lrcorner \varphi))$ because $(\alpha \to \neg\beta) \to \neg(\alpha \wedge \beta)$ is a theorem in intuitionistic logic.

Proposition 17

$P(\varphi, \neg \lrcorner \varphi) \to P(\neg \lrcorner \varphi, \varphi) := \mathsf{A}(\varphi \to \neg \lrcorner \varphi) \to \mathsf{A}(\neg \lrcorner \varphi \to \varphi)$.

1. $\vdash (\neg \lrcorner \varphi \to \varphi)$ from item 11 of Proposition 2.

2. $\vdash \mathsf{A}(\neg \lrcorner \varphi \to \varphi)$ by necessitation rule.

3. $\vdash \mathsf{A}(\neg \lrcorner \varphi \to \varphi) \to (\mathsf{A}(\varphi \to \neg \lrcorner \varphi) \to \mathsf{A}(\neg \lrcorner \varphi \to \varphi))$ by `A0`.

4. $\vdash (\mathsf{A}(\varphi \to \neg \lrcorner \varphi) \to \mathsf{A}(\neg \lrcorner \varphi \to \varphi))$ by `MP`.

Proposition 20: $\mathsf{E}(\blacklozenge \Diamond\varphi \wedge \psi) \leftrightarrow \mathsf{E}(\varphi \wedge \blacklozenge \Diamond\psi))$.

1. $\vdash \mathsf{A}\,\neg(\blacklozenge \Diamond\varphi \wedge \psi) \leftrightarrow \mathsf{A}(\blacklozenge \Diamond\varphi \to \neg\psi)$ by intuitionistic logic.

2.$\vdash \mathsf{A}(\blacklozenge \Diamond\varphi \to \neg\psi) \leftrightarrow \mathsf{A}(\Diamond\varphi \to \Box\neg\psi)$ by adjunction between $\blacklozenge$ and $\Box$.

3. $\vdash \mathsf{A}(\Diamond\varphi \to \Box\neg\psi) \leftrightarrow \mathsf{A}(\varphi \to \blacksquare\Box\neg\psi)$ by adjunction between $\Diamond$ and $\blacksquare$.

4. $\vdash \mathsf{A}(\varphi \to \blacksquare\Box\neg\psi) \leftrightarrow \mathsf{A}(\varphi \to \neg\blacklozenge \lrcorner \Box\neg\psi)$ by $\blacksquare p \leftrightarrow \neg\blacklozenge \lrcorner p$.

5. $\vdash \mathsf{A}(\varphi \to \neg\blacklozenge \lrcorner \Box\neg\psi) \leftrightarrow \mathsf{A}(\varphi \to \neg\blacklozenge \Diamond\psi)$ by $\Diamond p \leftrightarrow \lrcorner \Box\neg p$.

6. $\mathsf{A}(\varphi \to \neg\blacklozenge \Diamond\psi) \leftrightarrow \mathsf{A}\,\neg(\varphi \wedge \blacklozenge \Diamond\psi)$ bu=y intuitionistic logic.

7. $\vdash \mathsf{A}\,\neg(\blacklozenge \Diamond\varphi \wedge \psi) \leftrightarrow \mathsf{A}\,\neg(\varphi \wedge \blacklozenge \Diamond\psi)$ by concatenation lines 1-6

8. $\vdash \neg\,\mathsf{A}\,\neg(\blacklozenge \Diamond\varphi \wedge \psi) \leftrightarrow \neg\,\mathsf{A}\,\neg(\varphi \wedge \blacklozenge \Diamond\psi)$ from line 7.

9. $\vdash \mathsf{E}(\blacklozenge \Diamond\varphi \wedge \psi) \leftrightarrow \mathsf{E}(\varphi \wedge \blacklozenge \Diamond\psi)$ by item 22 of proposition 2.

C Proof of Theorem 4

Before giving a proof, we 'lift' our semantics based on an H-model for formulae to tableau expressions. As already explained in [19], the semantics of tableau expressions and the corresponding relation of satisfaction, $\Vdash$, is defined by an H-model (M, ι) with an assignment ι, where $M = (U, H, R, V)$ is an H-model and an assignment $\iota : \mathsf{Label} \to U$ is a function mapping labels of the tableau language to elements $w \in U$. Satisfaction of tableau expressions is defined as follows:

$$\begin{array}{l} M, \iota \nVdash \bot, \\ M, \iota \Vdash s : T\varphi \text{ iff } M, \iota(s) \models \varphi, \\ M, \iota \Vdash s : F\varphi \text{ iff } M, \iota(s) \not\models \varphi, \\ M, \iota \Vdash sHt \quad \text{iff } \iota(s)H\iota(t), \\ M, \iota \Vdash sRt \quad \text{iff } \iota(s)R\iota(t), \\ M, \iota \Vdash s \approx t \;\; \text{iff } \iota(s) = \iota(t), \\ M, \iota \Vdash s \not\approx t \;\; \text{iff } \iota(s) \neq \iota(t). \end{array}$$

Let us say that a set of tableau expression is *satisfiable* if there exists an H-model $M = (U, H, R, V)$ and an assignment $\iota : \mathsf{Label} \to U$ such that all the tableau expressions are satisfied in the pair (M, ι).

Now we proceed to our proof of Theorem 4.

Proof. Suppose that φ is provable from Γ. That is, the input set: $\{a : T\Gamma\} \cup \{a : F\varphi\}$ has a closed tableau. Let us fix this closed tableau and let us suppose by contradiction that $\Gamma \not\models \varphi$, so that φ is not a semantic consequence of the set Γ. So there is an H-model $M = (U, H, R, V)$ and state $w \in U$ such tha $M, w \models \Gamma$ and $M, w \not\models \varphi$. Consider an assignment function ι such that $\iota(a) = w$, then $M, \iota(a) \models \Gamma$ and $M, \iota(a) \not\models \varphi$. But then, under the assumption that the

Table 3. Tableau calculus Tab**UBiSKt**

$$\frac{s : T\varphi \quad s : F\varphi}{\bot}\ \text{closure}$$

$$\frac{s : T\varphi \wedge \psi}{s : T\varphi \quad s : T\psi}\ (T\wedge)$$

$$\frac{s : F\varphi \wedge \psi}{s : F\varphi \quad s : F\psi}\ (F\vee)$$

$$\frac{s : T\neg\varphi, s\ H\ t}{t : F\varphi}\ (T\neg)$$

$$\frac{s : F \lrcorner\, \varphi \quad t\ H\ s}{t : T\varphi}\ (F \lrcorner)$$

$$\frac{s : T\varphi \to \psi \quad s\ H\ t}{t : F\varphi \mid t : T\psi}\ (T \to)$$

$$\frac{s : F\varphi \prec \psi \quad t\ H\ s}{t : F\varphi \mid t : T\psi}\ (F\prec)$$

$$\frac{s : T\Box\varphi \quad s\ R\ t}{t : T\varphi}\ (T\Box)$$

$$\frac{s : T\blacklozenge\varphi \quad t\ R\ s}{t : F\varphi}\ (F\blacklozenge)$$

$$\frac{s : F\Diamond\varphi \quad t\ H\ s \quad t\ R\ u \quad v\ H\ u}{v : F\varphi}\ (F\Diamond),$$

$$\frac{s : T\blacksquare\varphi \quad s\ H\ t \quad u\ R\ t \quad u\ H\ v}{v : T\varphi}\ (T\blacksquare)$$

$$\frac{s : T\,\mathsf{A}\,\varphi, \quad t : S\psi}{t : T\varphi}\ (T\,\mathsf{A})$$

$$\frac{s : T\,\mathsf{E}\,\varphi}{m : T\varphi}\ (T\,\mathsf{E}) \text{ with } m \text{ fresh in the branch}$$

$$\frac{s : S\varphi}{s\ H\ s}\ \texttt{refl-H}$$

$$\frac{s : T\varphi \quad s\ H\ t}{t : T\varphi}\ \texttt{mon-H}$$

$$\frac{s : T(\bot)}{\bot}\ (T\bot)$$

$$\frac{s : F(\varphi \wedge \psi)}{s : F\varphi \mid s : F\psi}\ ((F\wedge)$$

$$\frac{s : T(\varphi \vee \psi)}{s : T\varphi \mid s : T\psi}\ ((T\vee)$$

$$\text{with } m \text{ fresh}\ \frac{s : F\neg\varphi}{s\ H\ m \quad m : T\varphi}\ (F\neg)$$

$$\text{with } m \text{ fresh}\ \frac{s : T \lrcorner\, \varphi}{m\ H\ s \quad m : F\varphi}\ (T \lrcorner)$$

$$\text{with } m \text{ fresh}\ \frac{s : F\varphi \to \psi}{s\ H\ m \quad m : T\varphi \quad m : F\psi}\ (F \to)$$

$$\text{with } m \text{ fresh}\ \frac{s : T\varphi \prec \psi}{m\ H\ s \quad m : T\varphi \quad m : F\psi}\ (T\prec)$$

$$\text{with } m \text{ fresh}\ \frac{s : F\Box\varphi}{s\ R\ m \quad m : F\varphi}\ (F\Box)$$

$$\text{with } m \text{ fresh}\ \frac{s : T\blacklozenge\varphi}{m\ R\ s \quad m : T\varphi}\ (T\blacklozenge)$$

$$\text{with } m, n, h \text{ fresh}\ \frac{s : T\Diamond\varphi}{m\ H\ s \quad m\ R\ n \quad h\ H\ n \quad h : T\varphi}\ (T\Diamond)$$

$$\text{with } m, n, h \text{ fresh}\ \frac{s : F\blacksquare\varphi}{s\ H\ m \quad n\ R\ m \quad n\ H\ h \quad h : F\varphi}\ (F\blacksquare)$$

$$\frac{s : F\,\mathsf{A}\,\varphi}{m : F\varphi}\ F\,\mathsf{A} \text{ with } m \text{ fresh in the branch}$$

$$\frac{s : F\,\mathsf{E}\,\varphi \quad t : S\psi}{t : F\varphi}\ F\,\mathsf{E}$$

$$\frac{s\ H\ t \quad t\ H\ j}{s\ H\ j}\ \texttt{trans-H}$$

$$\frac{s\ H\ t \quad t\ R\ j \quad j\ H\ k}{s\ R\ k}\ \texttt{stab-R}$$

rules of the calculus preserve satisfiablity, there should be at least one open branch for the set of labelled expressions $\{a : T\Gamma\} \cup \{a : F\varphi\}$. This is impossible by the initial assumption that $\{a : T\Gamma\} \cup \{a : F\varphi\}$ has a closed tableau. So the proof of soundness boils down to show that the rules of the calculus preserve satisfiability, that is, if the premise of a rule is satisfiable so is at least one of its conclusions. Since **TabBiSKt** has already been proved sound in [19], we focus only on the new rules $T\,\mathsf{A}$, $F\,\mathsf{A}$, since our arguments for $T\,\mathsf{E}$ and $F\,\mathsf{E}$ are similarly shown.

i) $T\,\mathsf{A}$: assume that the premise of the rule is satisfiable, so for some H-model and for some assignment ι we have $M, \iota(s) \models \mathsf{A}\,\varphi$ and $M, \iota(t) \models^* \psi$ where $\models^*$ is either $\models$ or $\not\models$. By the former, we have that $M, u \models \varphi$ for all $u \in U$. But $\iota(t) \in U$. So $M, \iota(t) \models \varphi$. So $M, \iota \Vdash t : T\varphi$ and an expanded branch with the rule's conclusion is satisfiable.

ii) $F\,\mathsf{A}$: assume that the premise of the rule is satisfiable, so for some H-model and for some assignment ι we have $M, \iota(s) \not\models \mathsf{A}\,\varphi$. But then there is some world $v \in W$ such that $M, v \not\models \varphi$. Fix such world v. Recall that m is a fresh label in the rule. We define a new assignment ρ by $\rho(m) = v$ and $\rho(x) = \iota(x)$ for all label $x \neq m$. Then it follows from $M, v \not\models \varphi$ that $M, \rho(m) \not\models \varphi$ that is $M, \rho \Vdash m : F\varphi$. If a tableau expression does not contain a fresh label m there is no difference between assignments ι and ρ. Therefore, an expanded branch with the rule's conclusion is satisfiable. □

D Proof of Theorem 5

Proof. The proof of equivalence of items 1 and 3 are due to the soundness and completeness results for **HUBiSKt**. So we need to show: i) if φ is a theorem in **HUBiSKt** then φ is a theorem in **TabUBiSKt**, and ii) if φ is a theorem in **TabUBiSKt** then φ is a theorem in **TabUBiSKt**. Proof of ii) follows from theorem 4 (soundness of **TabUBiSKt**) and theorem 2 (completeness of **HUBiSKt**). So we focus on i) here.

Proof of i): Recall that φ is a theorem **HUBiSKt** when φ follows from a set of axioms and rules given in Table 1. To show this direction, we reformulate our Hilbert-system into an equipollent system in the following two respects. First of all, to avoid the rule of uniform substitutions, we formulare our system in terms of axioms *schemes*. Second, we reformulate an inference rule 1 into an axiom as follows: for a rule of the form "from φ infer ψ", we can derive a formula $\mathsf{A}\,\varphi \to \mathsf{A}\,\psi$. Instead of showing an inference rule "from φ infer ψ" preserves theoremhood in **TabUBiSKt**, we show that $\mathsf{A}\,\varphi \to \mathsf{A}\,\psi$ is a theorem of **TabUBiSKt**. Then we can show that all the axiom schemes constructed from Table 1 are theorems in **TabUBiSKt** and that the above derived formula from an inference rule of Table 1 is a theorem in **TabUBiSKt**. We give here our proof for `Mon` $\blacklozenge$ as an example. The rule states that if $(p \to q)$ is a theorem then $(\blacklozenge\varphi \to \blacklozenge\psi)$ is a theorem. That means that the following implication must be a theorem: $\mathsf{A}(\varphi \to \psi) \to \mathsf{A}(\blacklozenge\varphi \to \blacklozenge\psi)$.

$$
\begin{array}{rcl}
 & s : F\,\mathsf{A}(\varphi \to \psi) \to \mathsf{A}(\blacklozenge\varphi \to \blacklozenge\psi) & \\ \hline
t \text{ fresh} & s\;H\;t \quad t : T\,\mathsf{A}(\varphi \to \psi) \quad t : F\,\mathsf{A}(\blacklozenge\varphi \to \blacklozenge\psi) & F\to \\ \hline
u \text{ fresh} & u : F(\blacklozenge\varphi \to \blacklozenge\psi) & F\,\mathsf{A} \\ \hline
v \text{ fresh} & u\;H\;v \quad v : T\blacklozenge\varphi \quad v : F\blacklozenge\psi & F\to \\ \hline
x \text{ fresh} & x\;R\;v \quad x : T\varphi & T\blacklozenge \\ \hline
 & x : F\psi & F\blacklozenge \\ \hline
 & x : T(\varphi \to \psi) & T\,\mathsf{A} \\ \hline
 & x\;H\;x & \texttt{refl-H} \\ \hline
 & x : F\varphi \mid x : T\psi & T\to \\ \hline
\text{closure} & \bot \mid \bot & \text{closure}
\end{array}
$$

The input set $\{s : F\,\mathsf{A}(\varphi \to \psi) \to \mathsf{A}(\blacklozenge\varphi \to \blacklozenge\psi)\}$ gives a closed tableau. Therefore the formula $\mathsf{A}(\varphi \to \psi) \to \mathsf{A}(\blacklozenge\varphi \to \blacklozenge\psi)$ is a theorem in Tab**UBiSKt**.

We also note that axioms and rules of H**UBiSKt** have been proved using our implementation of Tab**UBiSKt** in terms of the idea above on our reformulation. This can be checked following the link at [17].

E Proof of Lemma 4

Let φ and ψ be formulae and $[\![\varphi]\!]_M$ and $[\![\psi]\!]_M$ the associated H-sets in a given model M. In what follows we omit the subscription M. $[\![\varphi]\!] \not\subseteq [\![\psi]\!]$ iff for some $u \in U$, $u \in [\![\varphi]\!]$ and $u \notin [\![\psi]\!]$. Since H is reflexive, uHu holds, hence there is a $v \in U$ such that vHu and $v \in [\![\varphi]\!]$ and $v \notin [\![\psi]\!]$. By definition 3 this means that $M, u \models \varphi \prec \psi$, hence $M \models \mathsf{E}(\varphi \prec \psi)$. On the other direction, $M \models \mathsf{E}(\varphi \prec \psi)$ iff for some $u \in U$ $M, u \models \varphi \prec \psi$. Hence there is a $v \in U$ such that vHu and $M, v \models \varphi$ and $M, v \not\models \psi$, that means that $v \in [\![\varphi]\!]$ and $v \notin [\![\psi]\!]$, for some $v \in U$. Therefore $[\![\varphi]\!] \not\subseteq [\![\psi]\!]$.